# SUMMARY OF PROFESSIONAL ACCOMPLISHMENTS

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## Diplomas and scientific degrees

2010 Bachelor in Mathematics (Summa Cum Laude), Cracow University of Technology
Thesis: *Finite dimensional real division algebras*
Advisor: dr Marcin Skrzyński

2012 MSc in Mathematics (Summa Cum Laude), Cracow University of Technology
Thesis: *Frobenius theorem on linear determinant preservers and Linear Preserver Problems*
Advisor: dr Marcin Skrzyński

2015 PhD in Mathematics (Summa Cum Laude), Pedagogical University of Krakow
Thesis: *Minkowski decompositions and degenerations of Okounkov bodies*
Advisor: Prof. dr hab. Tomasz Szemberg
Auxiliary Advisor: dr hab. Jarosław Buczyński

## Employment history

2014–2015 *Research Assistant*, Institute of Mathematics, Pedagogical University of Krakow.
2015–2017 *Assistant Professor*, Institute of Mathematics, Pedagogical University of Krakow (on leave).
2015–2016 *Post-doc Position*, Institute of Mathematics, *Johannes Gutenberg Universität Mainz*.
2016–2017 *Post-doc Position and Privatdozent*, Institute of Algebraic Geometry, Leibniz Universität Hannover.
2017–2019 *Assistant Professor*, Institute of Mathematics of the Polish Academy of Sciences.
2019–2021 *Assistant Professor*, Institute of Mathematics, Pedagogical University of Krakow.
2021–still *Associate Professor*, Institute of Mathematics, Pedagogical University of Krakow.

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## Habilitation achievement

Singular curves in algebraic surfaces - algebra, geometry, and combinatorics.

[Hab1] Th. Bauer, P. Pokora, D. Schmitz, On the boundedness of the denominators in the Zariski decomposition on surfaces. *Journal für reine und angewandte Mathematik* **733**: 251 – 259 (2017).

IF: 1.686, MEiN points: 200,

[Hab2] P. Pokora, X. Roulleau, T. Szemberg, Bounded negativity, Harbourne constants and transversal arrangements of curves. *Annales de l'Institut Fourier Grenoble* **67(6)**: 2719 – 2735 (2017).

IF: 0.638, MEiN points: 140

[Hab3] R. Laface, P. Pokora, On the local negativity of surfaces with numerically trivial canonical class. *Rendiconti Lincei – Matematica e Applicazioni* **29**: 237 – 253 (2018).

IF: 0.690, MEiN points: 100

[Hab4] Piotr Pokora, J. Roé, Harbourne constants under ramified morphisms. *Results in Mathematics* **74(3)**: Article #109 - 24 pages (2019).

IF: 1.162, MEiN points: 100

[Hab5] P. Pokora, Hirzebruch-type inequalities viewed as tools in combinatorics. *Electronic Journal of Combinatorics* **28(1)**: #P1.9 - 22 pages (2021).

IF: 0.690, MEiN points: 100

[Hab6] A. Dimca, P. Pokora, On conic-line arrangements with nodes, tacnodes, and ordinary triple points. *Journal of Algebraic Combinatorics* **56(2)**: 403 – 424 (2022).

IF: 0.963, MEiN points: 100

[Hab7] P. Pokora, T. Szemberg, Conic-line arrangements in the complex projective plane. *Discrete and Computational Geometry*, Electronically Avaliable `doi:10.1007/s00454-022-00397-6` - 18 pages.

IF: 0.639, MEiN points: 100

## Description

### 1. Introduction

My research is devoted to the theory of curves in algebraic surfaces and I am working on three important problems in this area, namely

- the bounded negativity conjecture,
- constructions of very singular curves in algebraic surfaces, especially in the complex projective plane,
- variations on Terao's freeness conjecture and the notion of free and nearly-free reduced plane curves.

Let us present a general introduction to the subjects indicated above - we try to keep an informal style in order to provide main motivations and ideas standing behind that. In the report, we will use standard notations, mostly captured from [10, 23]. We work primarily over the complex numbers - some results and definitions can be formulated for an arbitrary field and we will emphasize this features directly in the text.

1.1. **Bounded Negativity.** We assume that every surface $X$ is normal and projective. We say that a curve $C \subset X$ is *negative* if $C$ is irreducible and reduced with $C^2 < 0$. The theory of curves in algebraic surfaces is a classical subject of algebraic geometry. This theory develops in many directions, one of them can be connected with the notion of the positivity (via Seshadri constants), but on the other hand, we still know very little about the *negativity* of algebraic curves. One of the most fundamental information about the way a curve is embedded into a surface is its self-intersection. At the beginning of the 20th century, the Italian School of Algebraic Geometry, probably in the person of F. Enriques, formulated the following conjecture, cf. [3].

We say that $X$ has *bounded negativity* if there exists $b(X) \in \mathbb{Z}$ such that for every irreducible and reduced curve $C$ one has $C^2 \geq -b(X)$.

**Conjecture 1.1.** *(BNC) Every surface $X$ in characteristic $0$ has bounded negativity.*

*Remark* 1. The BNC is obviously false in positive characteristic. Consider $X = C \times C$, where $C$ is a curve of genus $g(C) \geq 2$ defined over a field of characteristic $p > 0$. Let $\Gamma_n$ be the graph in $X$ of the $n$-th Frobenius morphism. We have

$$\Gamma_n^2 = p^n(2 - 2g(C))$$

and since $n$ can be arbitrarily large, then $X$ does not have bounded negativity.

It is known that some simple types of surfaces have bounded negativity.

**Proposition 1.2.** *(*[18, Corollary 1.2.3]*) A surface $X$ has bounded negativity if $-mK_X$ is effective for some positive integer $m$.*

In particular, we know that the blow-up of the complex projective plane $X_r$ along $r \in \{1, ..., 9\}$ (very) general points has the bounded negativity property - all negative curve are exactly $(-1)$-curves. However, it is not clear that the blowing up of $\mathbb{P}^2_{\mathbb{C}}$ along $10$ (very) general points has the bounded negativity property. It is worth emphasizing right now that if we do not require that configurations of points are (very) general, then the problem is getting more and more complicated, but also very interesting. Consider now an arrangement of $5$ general lines in $\mathbb{P}^2_{\mathbb{C}}$, these lines intersect along $10$ double intersection points. If we take the blowing up $X'_{10}$ of $\mathbb{P}^2_{\mathbb{C}}$ along 10 double intersection points, then $-K_{X'_{10}}$ is big. Recall that if $X$ is a complex rational surface such that $-K_X$ is big, then $X$ is a Mori Dream Surface [35]. In particular, Mori Dream Surfaces have only finitely many negative curves, and this is obviously our case with $X'_{10}$.

We have several natural questions regarding the bounded negativity, except the obvious one whether the bounded negativity conjecture holds. One of them is about its birationality.

**Question 1.3.** *Let $X$ and $Y$ be birationally equivalent projective surfaces. Does the BNC hold for $X$ if and only if it holds for $Y$? In other words: is the bounded negativity property a birational invariant?*

Let us remark here that the solution to the above problem, in characteristic zero, is not known even if $Y$ is the blow-up of $X$ at a single point. In positive characteristic, by a recent paper by Cheng and van Dobben de Bruyn published in Crelle's journal [8], we know that there exists an explicit sequence of blow-ups of the projective plane in positive characteristic that contain smooth rational curves of arbitrarily negative self-intersection. Due to this reason, the bounded negativity does not hold in positive characteristic even for rational surfaces and this gives us also the negative answer to the above question. However, both the bounded negativity conjecture and its birational invariance is still an open problem in characteristic zero.

The first problem that we have to face is to find an accurate way to measure the negativity for surfaces. Observe that we can easily (and artificially) construct very singular curves on blow-ups of surfaces. For example, we can take sufficiently many $s$ distinct points on a smooth curve $C$ of degree $d$ in $\mathbb{P}^2_{\mathbb{C}}$, then its strict transform $\tilde{C}$ under the blowing up along the $s$ points has the self-intersection equal to $d^2 - s$, so for sufficiently many points this self-intersection number can be very negative. In order to avoid such situations for blow-ups of surfaces, it is natural to study the weighted self-intersection numbers, i.e., we take the self-intersection of a given curve and divide this number by the number of points in which we blown up our surface. This idea is quite natural since the weighted self-intersections can be considered as a measure of the negativity with respect to the growth of the Picard number of the resulting surface. This observation stands behind the notions of the Harbourne constants and indices that allow to measure the local negativity of algebraic surfaces. It is worth noticing that some authors and referees of articles suggest to call these constants as *Hadean* due to the fact that it is very difficult to compute them, even for certain (sub)classes of curves in the complex projective plane. Here we are going to define and study Harbourne indices, mostly in order to keep the coherence of the results obtained by the applicant with co-authors or by other authors. This notion is mostly motivated by the curve viewpoint since it suggests focusing on singular loci rather than on arbitrary sets of points.

Let us recall that since the BNC holds for *reduced* curves in $X$ if and only if the BNC holds for *irreducible* and *reduced* curves [2, Proposition 3.8.2], we will mostly study the case of (reduced) curve arrangements.

From now on we assume that if $C \subset X$ is a singular curve, then by $\mathrm{Sing}(C)$ we denote the set of singular points of $C$.

**Definition 1.4.** *(Harbourne indices) Let $X$ be a smooth projective surface and $C \subset X$ a reduced curve. Then the Harbourne index of $C$ is defined as*

$$h(X;C) = \frac{C^2 - \sum_{P\in \mathrm{Sing}(C)} \mathrm{m}_P^2(C)}{s},$$

*where $s$ is equal to the cardinality of* $\mathrm{Sing}(C)$ *and* $\mathrm{m}_P(\cdot)$ *denotes the multiplicity of $C$ at $P \in \mathrm{Sing}(C)$. If $C \subset X$ is a reduced curve without singular points, then $h(X;C) = C^2$.*

*Similarly, we define* the global Harbourne index of X *as*

$$h(X) = \inf_C h(X, C),$$

*where the infimum is taken over all* reduced *curves $C \subset X$.*

Harbourne indices measure the local negativity of curves in surfaces. In particular, these indices allow to conclude that for certain classes of curves the BNC holds on blow-ups at singular loci of reduced curves. The above considerations present a new approach to attack the BNC for blow-ups from an asymptotical viewpoint. Concluding this section, let us present an extreme irreducible example.

*Example* 1. A result due to Severi [31] tells us that there exists an irreducible rational curve $C_d \subset \mathbb{P}^2_{\mathbb{C}}$ of degree $d$ with $g := \frac{(d-1)(d-2)}{2}$ nodes. Then it follows that

$$h(\mathbb{P}^2_{\mathbb{C}}; C_d) = \frac{d^2 - 4 \cdot g}{g} = 2\left(\frac{-d^2 + 6d - 2}{d^2 - 3d + 2}\right) \xrightarrow{d \to \infty} -2.$$

This example shows that we always have $h(\mathbb{P}^2_{\mathbb{C}}) \leq -2$. In this report we present results providing lower-bounds on Harbourne indices for various classes of surfaces and curves. In particular, we explain the case of curve arrangements in the complex projective plane, assuming that our curves have ordinary singularities, and the case of rational curves in surfaces having trivial canonical class.

1.2. **Highly singular curves.** The theory of plane curve is a very classical subject which dates back to the beginnings of the projective geometry. If we think about curves and their combinatorial properties, our first try is to approach line arrangements in the projective plane over a given field $\mathbb{F}$. The first problem that we need to face is to collect combinatorial constraints that can potentially allow us to exclude the existence of certain *combinatorial configurations*. For instance, we know that if $\mathcal{L} \subset \mathbb{P}^2$ is an arrangement of $d$ lines, then the following combinatorial count holds:

$$\binom{d}{2} = \sum_{r \geq 2} \binom{r}{2} t_r,$$

where $t_r$ denotes the number of $r$-fold intersection points of lines. If we take $d = 7$, then by this Diophantine equation we can potentially have $t_3 = 7$. It is natural to wonder whether we can construct such a configuration of points and lines in the complex projective plane. It turns out, and it was discovered years ago, that such a configuration exists if the underlying field has characteristic 2, and if we take $\mathbb{F} = \mathbb{Z}_2$ we obtain the famous Fano plane. Another question that we can ask, and if fact this is an **open problem**, whether there exists an arrangement of $d = 13$ lines with $t_3 = 26$. We predict that such a configuration of points and lines cannot exist, but we do not have a formal proof yet! Based on that small sample, it is clear that we want to find the tightest possible constraints on the combinatorics of line arrangements in order to decide whether a certain combinatorial data provided by the above Diophantine equation can be geometrically realized over a given field. In the case of line arrangements, Hirzebruch in his famous paper devoted to ball-quotient surfaces [19] provides, **as a simple by-product**, the following famous inequality.

**Theorem 1.5** (Hirzebruch)**.** *Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $d \geq 6$ lines such that $t_d = t_{d-1} = 0$. Then one has*

$$t_2 + t_3 \geq d + \sum_{r \geq 5} (r-4)t_r.$$

This result is powerful in the context of many applications, especially in the so-called extreme problems in the geometry of points and lines. On the other hand, using only the above inequality, we are not able to exclude the existence of a configuration with $d = 13$ lines and $26$ triple points. This observation shows a new branch of studies, namely how to construct reduced plane curves having prescribed singularities. It turns our that this problem is extremely difficult and this will be visible in the course of the present document.

In our research we focus on both aspects of the game, namely we prove the so-called *Hirzebruch-type inequalities* that allow us to find combinatorial constraints on certain classes of reduced curves in surfaces. On the other hand, we also use certain methods to construct new examples of highly singular curves. It is very natural to use symmetries - for instance the geometry standing behind irreducible complex reflection groups, of robust properties of ramified morphisms. We will present outcomes coming from these two approaches constructing highly singular curves with very interesting properties, especially in the framework of the bounded negativity conjecture.

1.3. **On Terao's Freeness Conjecture.** Here we are going to present a short introduction to the subject using a very classical approach. Let $S := \mathbb{C}[x,y,z] = \bigoplus_k S_k$ be the graded polynomial ring and let $f \in S$ be a homogeneous polynomial of degree $d$. Consider a curve $C \,:\, f = 0$ in $\mathbb{P}^2_{\mathbb{C}}$ defined by $f$. We start our discussion looking at the most classical notation, namely we focus on **free curves**. Denote by $\partial_x, \partial_y, \partial_z$ the partial derivatives and we define $\mathrm{Der}(S) = \{\partial := a \cdot \partial_x + b \cdot \partial_y + c \cdot \partial_z,\ a,b,c \in S\}$ the free $S$-module of $\mathbb{C}$-linear derivations of the ring $S$. Now for a reduced curve $C \,:\, f = 0$ with $f \in S_d$ being homogeneous, we define

$$\mathrm{D}(f) = \{\partial \in \mathrm{Der}(S) \,:\, \partial\, f \in \langle f \rangle\}.$$

It means that $\mathrm{D}(f)$ is the graded $S$-module of derivations preserving the ideal $\langle f \rangle$. We can show that for a reduced plane curve $C \,:\, f = 0$ in $\mathbb{P}^2_{\mathbb{C}}$ we have the following decomposition:

$$\mathrm{D}(f) = \mathrm{D}_0(f) \oplus S \cdot \delta_E,$$

where $\delta_E = x\partial_x + y\partial_y + z\partial_z$ is the Euler derivation and

$$\mathrm{D}_0(f) = \{\partial \in \mathrm{Der}(S) \,:\, \partial\, f = 0\},$$

i.e., it is the set of all $\mathbb{C}$-linear derivations of $S$ killing the polynomial $f$.

**Definition 1.6.** *We say that a reduced curve $C \,:\, f = 0$ in $\mathbb{P}^2_{\mathbb{C}}$ defined by a homogeneous polynomial $f \in S$ is **free** if $\mathrm{D}(f)$ (or just $\mathrm{D}_0(f)$) is a free graded $S$-module.*

For a homogeneous polynomial $g \in S$ of degree $d$ we define its Jacobian ideal $J_g := \langle \partial_x\, g, \partial_y\, g, \partial_z\, g \rangle$, and we define by $I_g$ the saturation of $J_g$ with respect to the irrelevant ideal $\mathfrak{m} = \langle x,y,z \rangle$. The Jacobian module of $g$ is defined as

$$N(g) = I_g / J_g.$$

The Jacobian module provides an important information about the curve that is associated with $g \in S$. One can show that the freeness of $C \,:\, g = 0$ boils down to the condition

$N(g) = 0$. We set $n(g)_j = \dim N(g)_j$, and for a reduced curve $C\,:\,g = 0$ in $\mathbb{P}^2_{\mathbb{C}}$ given by $g \in S$ we define the following invariants:

$$\sigma(C) = \min\{j\,:\,n(g)_j \neq 0\} \text{ and } \nu(C) = \max\{n(g)_j\}_j.$$

The invariant $\nu(C)$ is called the **defect**, or the freeness defect. Using the notion of the defect, we are ready to introduce the second most important class of plane curves that we are going to discuss here.

**Definition 1.7.** *A reduced curve $C\,:\,g = 0$ in $\mathbb{P}^2_{\mathbb{C}}$ defined by a homogeneous polynomial $g \in S$ is **nearly-free** if $\nu(C) = 1$.*

The class of free and nearly-free curves plays an important role in the context of the Saito-Terao's freeness conjecture that is formulated, generally, for hyperplane arrangements in projective spaces. Here we present the planar version and due to this reason we assume here that $\mathcal{A}$ is an arrangement of $d$ lines in $\mathbb{P}^2_{\mathbb{C}}$. Denote by $L(\mathcal{A})$ its intersection lattice which is the set of all flats, i.e., non-empty intersections of (sub)families of lines in $\mathcal{A}$, with the order defined by $X \leq Y$ if and only if $Y \subset X$. The intersection lattice is the most fundamental object that can be associated with $\mathcal{A}$ and it decodes the whole combinatorial information attached to $\mathcal{A}$. In the 1980s, Terao in [36] asked whether the intersection lattice for line arrangements (or in general, for hyperplane arrangements) determines the property of being free.

**Conjecture 1.8** (Terao)**.** *Let $\mathcal{A}, \mathcal{B} \subset \mathbb{P}^2_{\mathbb{C}}$ be two line arrangements such that their intersection lattices $L(\mathcal{A}), L(\mathcal{B})$ are isomorphic. Assume that $\mathcal{A}$ is free, then $\mathcal{B}$ must be free.*

Notice that Terao's freeness conjecture is widely open - we know that it holds with up to $14$ lines [1], which might be disappointing. However, Terao's conjecture is very demanding and in order to understand this problem well one needs to understand many aspects, for instance the geometry of moduli spaces of line arrangements which is not easy at all. It is very unclear whether the mentioned conjecture holds in general, and based on that problem Dimca and Sticlaru in [11] defined the class of nearly-free curves and shortly afterwards it turned out that nearly-free curves might be crucial for Terao's conjecture. It is believed that if there exists a counterexample to Terao's freeness conjecture, then we should be able to find two line arrangements with isomorphic intersection lattices, where one arrangement is free and the second arrangement is nearly-free. This prediction explains why free and nearly-free curves gain a lot of attention of researchers working on plane curves. It is worth pointing out here that there are some attempts to generalize Terao's freeness conjecture to other classes of reduced plane curves. It turned out that naively generalized Terao's freeness conjecture to conic-line arrangements in $\mathbb{P}^2_{\mathbb{C}}$ is false [32]. Let us present here this (counter)example as our main motivation for further discussion in the section devoted to habilitation achievements.

*Example* 2. Consider the following conic-line arrangement

$$\mathcal{CL}_1\,:\,xy \cdot (y^2 + xz) \cdot (y^2 + x^2 + 2xz) = 0.$$

The intersection point $P = (0 : 0 : 1)$ has multiplicity $4$ and it is quasi-homogeneous (although it is not **ordinary**). One can show that $\mathcal{CL}_1$ is free. If we perturb a bit line $y = 0$, taking for instance $x - 13y = 0$, we obtain a new conic-line arrangement

$$\mathcal{CL}_2\,:\,x \cdot (x - 13y) \cdot (y^2 + xz) \cdot (y^2 + x^2 + 2xz) = 0.$$

In this new arrangement, the intersection point $P = (0 : 0 : 1)$ has multiplicity $4$, but it is not longer quasi-homogeneous, and $\mathcal{CL}_2$ is not free. In fact, the arrangement $\mathcal{CL}_2$ is nearly-free, as defined above.

Based on results and examples presented by Schenck and Tohaneanu in [32], our problem boils down to understand a weaker version of the Terao's freeness conjecture that we are going to formulate here. In order to do so, we need to introduce the weak combinatorics of a given plane curve $C$ in $\mathbb{P}^2_{\mathbb{C}}$ which is adequate for our setting.

**Definition 1.9.** *Let $C = \{C_1, ..., C_k\} \subset \mathbb{P}^2_{\mathbb{C}}$ be a reduced curve such that each irreducible component $C_i$ is **smooth**. The weak combinatorics of $C$ is a vector $(d_1, ..., d_s; t_1, ..., t_p)$, where $d_i$ denotes the number of all irreducible components of $C$ having degree $i$, and $t_j$ denotes the number of singular points of a curve $C$ of a given topological type $S_j$.*

For instance, if $\mathcal{A} \subset \mathbb{P}^2_{\mathbb{C}}$ is a line arrangement, then the weak combinatorics of $\mathcal{A}$ is $(d, t_2, ..., t_d)$, where $d$ is the number of lines and $t_j$ denotes the number of $j$-fold intersection points.

**Conjecture 1.10** (Numerical Terao's Conjecture)**.** *Let $C_1, C_2$ be two reduced curves in $\mathbb{P}^2_{\mathbb{C}}$ such that all irreducible components of them are smooth. Suppose that $C_1$ and $C_2$ have the same weak combinatorics and let $C_1$ be free, then $C_2$ is also free.*

Our main result into this direction shows that Numerical Terao's Conjecture holds for complex conic-line arrangements in the plane admitting nodes, tacnodes, and ordinary triple points. We will present details in the forthcoming sections.

## 2. Habilitation achievement

2.1. **The bounded negativity conjecture versus Zariski decompositions.** In this section, we present the main result of the report, namely we show that the Bounded Negativity Conjecture is decoded by Zariski decompositions of pseudo-effective integral divisors (i.e., divisors with non-negative integer coefficients), and vice versa [Hab1]. This result is highly unexpected since it provides a direct linkage between the negativity of curves and numerical properties of the Zariski decomposition for surface.

The theorem on Zariski decomposition is a fundamental tool in the theory of algebraic surfaces. It was established by Zariski [38] for effective divisors and extended by Fujita [15] to the pseudo-effective case. The geometric significance of Zariski decompositions lies in the fact that, given a pseudo-effective integral divisor $D$ on $X$ with the Zariski decomposition $D = P + N$, one has for every sufficiently divisible integer $m \geq 1$ the equality

$$H^0(X, \mathcal{O}_X(mD)) = H^0(X, \mathcal{O}_X(mP)) .$$

In other words, all sections of $\mathcal{O}_X(mD)$ come from the nef line bundle $\mathcal{O}_X(mP)$. The term *sufficiently divisible* here means that one needs to pass to a multiple $mD$ that clears denominators in $P$ for the statement to hold. Of course, it would be most pleasant if one knew – beforehand and independently of $D$ – which multiple to take. This amounts to asking the following.

**Problem 2.1.** Let $X$ be a smooth projective surface. Does there exist an integer $d(X) \geq 1$ such that for every pseudo-effective integral divisor $D$ the denominators in the Zariski decomposition of $D$ are bounded from above by $d(X)$?

If such a bound $d(X)$ exists, then we say that $X$ *has bounded Zariski denominators*.

**Conjecture 2.2.** *(The Bounded Denominators Conjecture) Every smooth projective surface $X$ in characteristic zero has bounded Zariski denominators.*

Let us point out here that this question was formulated by Alex Küronya in 2009, and this problem was open until 2017.

Taking then the factorial $d(X)!$, one has in fact a uniform number that clears denominators in all Zariski decompositions on $X$. It is an intriguing question as to whether a given smooth surface satisfies this boundedness condition.

It turned out, somewhat surprisingly, that the boundedness of Zariski denominators is equivalent to the bounded negativity property.

**Theorem 2.3** ([Hab1]). *For a smooth projective surface $X$ over an algebraically closed field the following two statements are equivalent:*

(i) *$X$ has bounded Zariski denominators,*
(ii) *$X$ has bounded negativity.*

In fact, one can find strict relations between numbers $d(X)$ and $b(X)$ for a given surface $X$.

**Denominators in Zariski decompositions**

Let $X$ be a smooth projective surface and $D$ a pseudo-effective integral divisor on $X$. Fujita's extension [15] of Zariski's result [38] states that $D$ can be written uniquely as a sum

$$D = P + N$$

of $\mathbb{Q}$-divisors such that

(i) $P$ is nef,
(ii) $N$ is effective and has negative definite intersection matrix if $N \neq 0$,
(iii) $P \cdot C = 0$ for every component of $N$.

For the question of bounded denominators in $P$ and $N$ it is of course enough to consider the denominators of $N = \sum_{i=1}^{k} a_i N_i$, i.e., the denominators of the coefficients $a_i$. In order to approach the problem, we use the following description of the coefficients $a_i$. They are given as the (unique) solution of the system of equations

$$D \cdot N_j = (P + \sum_{i=1}^{k} a_i N_i) \cdot N_j = \sum_{i=1}^{k} a_i N_i \cdot N_j \qquad \text{for all } j \in \{1, \ldots, k\}.$$

This system can be rewritten in matrix form as

$$S[a_1, \ldots, a_k]^t = [D \cdot N_1, \ldots, D \cdot N_k]^t \,,$$

where $S$ denotes the intersection matrix of the curves $N_1, \ldots, N_k$, i.e., $S = [N_i \cdot N_j]_{i,j} \in M_{k \times k}(\mathbb{Z})$. Since the matrix $S$ is negative definite, it has non-zero determinant, and using Cramer's rule one has

$$a_i = \frac{\det[s_1, \ldots, s_{i-1}, b, s_{i+1}, \ldots, s_k]}{\det(S)} \,, \tag{1}$$

where $s_i$ denotes the $i$-th column of the matrix $S$ and $b = [D \cdot N_1, \ldots, D \cdot N_k]^t$. Thus, for divisors with negative part $N$ supported on $N_1, \ldots, N_k$, the denominators of the Zariski decomposition are bounded by $|\det(S)|$.

*Remark* 2. Note that the above reasoning yields an upper bound for the denominators of the coefficients in the Zariski decomposition for any surface whose pseudoeffective cone is rational polyhedral, since in this case there are only finitely many possible sets $\{N_1, \dots, N_k\}$ of components of negative parts, so we obtain the bound

$$d(X) = \max\{|\det(S_i)| \mid S_i \text{ principal negative definite submatrix of } S\},$$

where $S$ denotes the intersection matrix of all irreducible curves with negative self-intersections. It is not clear a priori whether the corresponding supremum will be finite in the presence of infinitely many extremal rays.

**Bounded denominators and bounded negativity**

The main aim here is to show key ingredients of Theorem 2.3. The results below show that it is possible to find relations between numbers $b(X)$ and $d(X)$ in a strict sense.

The first result provides a bound for $d(X)$ using only information about $b(X)$ and the Hodge Index Theorem.

**Theorem 2.4.** *Let $X$ be a smooth projective surface on which the self-intersection of irreducible curves is bounded by $-b(X)$. Then $X$ has bounded Zariski denominators. More concretely, denoting by $\rho(X)$ the Picard number, we have*

$$d(X) \le b(X)^{\rho(X)-1}.$$

We now turn to the converse implication.

**Theorem 2.5.** *Let $X$ be a smooth projective surface. If Zariski denominators on $X$ are bounded by $d(X)$, then $X$ has bounded negativity. More concretely, denoting by $\Delta$ the discriminant of the Néron-Severi lattice $N^1(X)$ (i.e., the determinant of the intersection form), we have*

$$b(X) \le d(X) \cdot d(X)! \cdot |\Delta|.$$

We present here several interesting examples of surfaces for which the Bounded Denominators Conjecture is true. However, we start with the example in positive characteristic with the property that neither the Bounded Negativity nor the Bounded Denominators Conjecture holds (via Theorem 2.3).

*Example* 3. (Surfaces with unbounded Zariski denominators in positive characteristic) Let $C$ be a curve of genus $g \ge 2$ defined over a finite field of characteristic $p > 0$. The surface $X = C \times C$ is then known to have unbounded negativity. Indeed, taking for $n \in \mathbb{N}$ the graph $\Gamma_n$ of the Frobenius morphism obtained by taking $p^n$-th powers, we have $\Gamma_n^2 = p^n(2-2g) \to -\infty$. By Theorem 2.5, $X$ must have unbounded Zariski denominators. In the particular case at hand, these are in fact quickly detected. Denote by $F_2$ a fiber of the second projection $X \to C$, and consider the divisor $D_n = F_2 + \Gamma_n$. The negative part of its Zariski decomposition has support $\Gamma_n$ with coefficient

$$\frac{D_n \cdot \Gamma_n}{\Gamma_n^2} = \frac{1 + \Gamma_n^2}{\Gamma_n^2}$$

Since the numerator and denominator are coprime for all $n$, we see that the Zariski denominator is $-\Gamma_n^2 = p^n(2g-2)$ and hence tends to infinity.

Next, we determine precise bounds on the Zariski denominators for classes of surfaces $X$ for which bounded negativity holds and explicit bounds $b(X)$ are known.

*Example* 4. (Surfaces with nef anticanonical bundle) Let $X$ be a smooth projective surface with $-K_X$ nef. As a consequence of the adjunction formula, we have the negativity bound

$b(X) = 2$. Indeed, for every irreducible curve one has $2p_a(C) - 2 = K_X \cdot C + C^2 \leq C^2$, and hence $C^2 \geq -2$.

It means that for every pseudo-effective integral divisor $D$ on $X$, the Zariski decomposition of $2^{\rho-1}! \cdot D$ is integral.

**Further developments regarding Denominators in Zariski decompositions**

Here we briefly discuss results that are motivated by [Hab1].

[1] B. Harbourne, P. Pokora, H. Tutaj-Gasińska, On integral Zariski decompositions of pseudoeffective divisors on algebraic surfaces. *Electron. Res. Announc. Math. Sci.* **22**: 103-108 (2015).

Our first problem that we tried to approach is the following.

**Problem 2.6.** *Let $X$ be a smooth complex projective surface such that $d(X) = 1$. Is it true that one always has $b(X) = 1$?*

The answer to this question is negative, namely there exists a smooth complex projective surface with $d(X) = 1$, but $b(X) = 2$.

**Theorem 2.7.** *There exists a smooth complex $K3$ surface $X$ of Picard number $2$ having intersection form*

$$\begin{pmatrix} -2 & 4 \\ 4 & -2 \end{pmatrix}$$

*such that all integral pseudoeffective divisors on $X$ have integral Zariski decompositions.*

We also obtained, somehow surprisingly, that there exists a strict relation between a weak version of the SHGH conjecture and integral Zariski decompositions of integral pseudoeffective divisors on blow-ups of the complex projective plane. Let us recall that a weak version of the SHGH conjecture tells us that on the blowing-up of the complex projective plane along $s \geq 10$ very general points the only negative curves are $(-1)$-curves.

**Theorem 2.8.** *Let $\pi : X \to \mathbb{P}^2$ be the blowing up (over an algebraically closed ground field $K$ of arbitrary characteristic) of a finite set of points $p_1, \ldots, p_s$ (possibly infinitely near). Suppose that every integral pseudoeffective divisor $D$ has an integral Zariski decomposition. Then all negative curves on $X$ have self-intersection $-1$ (i.e., are $(-1)$-curves).*

[2] M. Kapustka, G. Mongardi, G. Pacienza, P. Pokora, On the Boucksom-Zariski decomposition for irreducible symplectic varieties and bounded negativity. Electronically available at **arXiv:1911.03367**.

In the paper we study a natural generalization of the boundedness problem for Zariski decompositions to varieties of dimension greater than 2. Our decision to study smooth projective irreducible symplectic varieties is motivated by the existence of the so-called Boucksom-Zariski decomposition that has all desired properties for studying boundedness questions.

**Theorem 2.9.** *Let $X$ be a smooth projective irreducible symplectic variety of Picard number $\rho(X)$. The denominators of the coefficients of the negative and positive parts of the Boucksom-Zariski decompositions of all pseudo-effective Cartier divisors are bounded by $(4 \cdot \mathrm{Cardinality}(A_X))^{\rho(X)-1})!$, where*

$$A_X := H^2(X, \mathbb{Z})^\vee / H^2(X, \mathbb{Z})$$

*is the finite discriminant group of the intersection lattice.*

**Corollary 2.10.** *Let $X$ be a smooth projective irreducible symplectic variety of dimension $2n$ and let $L \in \mathrm{Pic}(X)$ be a big line bundle. Then for all*

$$m \geq \frac{1}{2}(2n+2)(2n+3)(4 \cdot \mathrm{Cardinality}(A_X))^{\rho(X)-1})!$$

*the map associated to the linear system $|mL|$ is birational onto its image.*

2.2. **Harbourne indices and curve configurations in algebraic surfaces.** At the very beginning, we focus on the so-called $d$-configurations of curves in the complex projective plane.

**Definition 2.11.** *Let $\mathcal{C} = \{C_1, ..., C_\tau\} \subset \mathbb{P}^2_{\mathbb{C}}$ be a configuration of $\tau \geq 3$ curves. We say that $\mathcal{C}$ is a $d$-configuration if*

- *all irreducible components $C_i$ are smooth and have the same fixed degree $d \geq 1$,*
- *every pair of curves intersects transversally, i.e., locally these intersections look like $x_1 \cdot x_2 = 0$,*
- *there is no point where all curves meet.*

Our $d$-configuration can be considered as a natural generalization of line configurations, i.e., $1$-configurations are exactly line arrangements in the plane. In particular, we will say that $2$-configurations are simply conic configurations even if it might happen that in general configurations of conics have non-ordinary singularities, but we hope that our abuse is not going to be misleading. Along the same lines, let $\mathcal{C}$ be a $d$-configuration of $\tau \geq 3$ curves (or any configuration of smooth irreducible curves having only ordinary singularities), then by $t_r$ we denote the number of $r$-fold points, i.e., points where exactly $r$ curves from $\mathcal{C}$ meet. Moreover, for $i \in \{0, 1, 2\}$, we define

$$f_i = \sum_{r \geq 2} r^i t_r.$$

**Definition 2.12.** *(Degree $d$-global Harbourne index) The* degree $d$-global Harbourne index of $\mathbb{P}^2_{\mathbb{C}}$ *is the infimum*

$$h_d(\mathbb{P}^2_{\mathbb{C}}) := \inf_{\mathcal{C}} h(\mathbb{P}^2_{\mathbb{C}}; \mathcal{C})$$

*taken over all $d$-configurations $\mathcal{C}$ with fixed $d \geq 1$ in $\mathbb{P}^2_{\mathbb{C}}$.*

We start with line arrangements in the complex projective plane. This result was presented by the applicant in 2014 during the workshop *Negative Curves on Algebraic Surfaces* [12, p. 567] and it was used to show the boundedness of the so-called global linear Harbourne constant in [4] - here we present a simplified version of this result in order to keep the coherence of the report.

**Theorem 2.13** ([4])**.** *With the above notation, one has*

$$h_1(\mathbb{P}^2_{\mathbb{C}}) \geq -4.$$

In order to show this lower-bound, we used a very strong result in the theory of complex line arrangements, namely Hirzebruch's inequality [19].

**Theorem 2.14.** *(Improved Hirzebruch's inequality) Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $\tau \geq 4$ lines such that $t_\tau = t_{\tau-1} = 0$. Then*

$$t_2 + \frac{3}{4}t_3 \geq \tau + \sum_{r \geq 5}(r-4)t_r.$$

Based on this result, it is quite easy to provide a lower bound on $h_1(\mathbb{P}^2_{\mathbb{C}})$ and we refer to [4] for the details. If we have a lower bound, which seems to be quite good, it is natural to wonder whether we have any example approaching the value $-4$. The following example is the world-record case for lowering the value of Harbourne indices for line arrangements in the complex projective plane.

*Example* 5. This construction dates back to Wiman's work [39] and it is strictly related to the theory of irreducible complex reflection groups. There exists the arrangement $\mathcal{W}$ (since it is unique up to the projective equivalence) of $45$ complex lines in the projective plane that they intersect at $120$ triple, $45$ quadruple and $36$ quintuple points. If we compute its Harbourne index, we obtain

$$h(\mathbb{P}^2_{\mathbb{C}}; \mathcal{W}) = \frac{45 - 3\cdot 120 - 4 \cdot 45 - 5 \cdot 36}{201} = -\frac{225}{67} \approx -3.36.$$

Now we want to report on the values of Harbourne indices for conic arrangements in the plane. This result was obtained by the applicant and H. Tutaj-Gasińska in [30].

**Theorem 2.15.** *One has*

$$h_2(\mathbb{P}^2_{\mathbb{C}}) \geq -\frac{9}{2}.$$

If we allow our conic arrangements $\mathcal{C}$ to have the property that either $t_\tau = 4$ or $t_\tau = 3$, then still the bound $-\frac{9}{2}$ holds for such arrangements. Moreover, if conic arrangements $\mathcal{C}$ have $t_\tau = 2$, then we find a lower bound on Harbourne indices for such arrangements, but our formula is rather complicated. In order to obtain such a bound, we used a natural relation between conic arrangements having two base points and arrangements of $(1,1)$-curves in the complex quadric $\mathbb{P}^1 \times \mathbb{P}^1$. In the case of $t_\tau = 1$, we cannot say anything, which is somehow disappointing, but this is a consequence of some technical obstructions, i.e., we cannot perform Hirzerbuch's construction in this case successfully.

Now we focus on the case of $d$-configurations with $d \geq 3$. The first step is the following Hirzebruch-type inequality proved by the applicant and this can be considered as the second most important scientific achievement in the portfolio of the applicant.

**Theorem 2.16** ([Hab2])**.** *Let $\mathcal{C} \subset \mathbb{P}^2_{\mathbb{C}}$ be a $d$-configuration with $d \geq 3$ and $\tau \geq 4$. Then*

$$\left(\frac{7}{2}d^2 - \frac{9}{2}d\right)\tau + t_2 + t_3 \geq \sum_{r\geq 5}(r-4)t_r.$$

Let us explain how to prove this result shortly in order to present methods involved in our proof. The key idea dates back to Hirzebruch in the late-eighties. He observed that one can use abelian covers of the complex projective plane branched along line arrangements. These abelian covers are given by the so-called Kummer extensions. We can mimic this construction and study abelian covers of the complex projective plane branched along a given

$d$-arrangement $\mathcal{C}$ with $d \geq 3$ having the exponent $n$. This covering leads us to an algebraic surface $X_n$ with normal singularities which ramifies over the arrangement. One can show, using local arguments on the surface, that $X_n$ is singular exactly over a point $p$ if and only if $p$ is a point of multiplicity $\geq 3$ in the arrangement. After blowing up these singular points we obtain a smooth surface $Y_n^{\mathcal{C}}$. It turns out that the Chern numbers of $Y_n^{\mathcal{C}}$ can be read off directly from the combinatorics of the arrangement $\mathcal{C}$. Moreover, it can be shown that $Y_n^{\mathcal{C}}$ has non-negative Kodaira dimension if $n \geq 2$, $d \geq 3$, and the number of irreducible components of $\mathcal{C}$ is grater than or equal to $4$. In these cases we can apply the Bogomolov-Miyaoka-Yau inequality [24]:

$$c_1^2(Y_n^{\mathcal{C}}) \leq 3c_2(Y_n^{\mathcal{C}}).$$

Now we can define the following Hirzebruch polynomial:

$$P_{\mathcal{C}}(n) = \frac{3c_2(Y_n^{\mathcal{C}}) - c_1^2(Y_n^{\mathcal{C}})}{n^{d-3}}.$$

By the Bogomolov-Miyaoka-Yau inequality we have $P_{\mathcal{C}}(n) \geq 0$, so plugging the value $n = 3$ to $P_{\mathcal{C}}(n)$ gives us the desired inequality. Even if this construction is quite technical and involving, the outcome is very handy and can be directly use to study questions revolving around the bounded negativity problem. By example, the above inequality allows to prove directly the following result.

**Theorem 2.17** ([Hab2])**.** *Let $d \geq 3$ be fixed. Then*

$$h_d(\mathbb{P}^2_{\mathbb{C}}) \geq -4 - \frac{5}{2}d^2 + \frac{9}{2}d.$$

Now we pass to another class of curves and surfaces. From now on we assume that $X$ is a smooth complex projective surface with trivial canonical class, for instance $X = K3$. We consider rational curve arrangements in $X$, i.e., $\mathcal{C} = \{C_1, ..., C_\tau\} \subset X$ is an arrangement of smooth irreducible rational curves such that all singularities are ordinary. The first main result in this new setting is a Hirzebruch-type inequality for rational curve arrangements, and this is the third most important result of the applicant in the context of bounding Harbourne indices.

**Theorem 2.18** ([Hab3])**.** *Let $X$ be a smooth complex projective surface with trivial canonical class. Assume that $\mathcal{C} \subset X$ is an arrangement of smooth irreducible rational curves having $\tau$ irreducible components and only ordinary singularities. Then*

$$4\tau - t_2 + \sum_{r\geq 3}(r-4)t_r \leq 3c_2(X) \leq 72.$$

Our proof of the above Hirzebruch-type inequality is based on the so-called logarithmic version of the Bogomolov-Miyaoka-Yau inequality [25] and combinatorial considerations for such curve arrangements. This result is a crucial step towards providing a lower bound on the following global Harbourne index.

**Definition 2.19.** *Let $X$ be a smooth complex projective surface having trivial canonical class. The real number*

$$H_{\text{rational}}(X) = \inf_{\mathcal{C}} h(X; \mathcal{C}),$$

*where the infimum is taken over all rational curve arrangements having only ordinary singularities $\mathcal{C} \subset X$, is called the global rational Harbourne index of $X$.*

**Theorem 2.20** ([Hab3])**.** *In the setting as above and under the assumptions as in Theorem 2.18, one has*

$$H_{\text{rational}}(X) \geq -45.$$

It is natural to wonder how negative Harbourne indices of our rational curve arrangements could be. We addressed this problem in the second part of [Hab3] – we found, for instance, two interesting arrangements in smooth hypersurfaces of degree $4$ in $\mathbb{P}^3_{\mathbb{C}}$ having exactly $16$ lines and $8$ quadruple points providing Harbourne indices equal to $-8$. This shows that starting with negative curves in algebraic surfaces we obtain a completely different picture of Harbourne indices comparing with the case of the complex projective plane.

### Further developments regarding Harbourne indices and curve arrangements

Here we briefly discuss **very selected results** that are related to the main core of $[\mathrm{Hab2}]$ and $[\mathrm{Hab3}]$ - the order below is chronological.

[1] X. Roulleau, Bounded Negativity, Miyaoka—Sakai Inequality, and Elliptic Curve Configurations. *Int. Math. Res. Not.* **2017(8)**: 2480–2496 (2017).

The author studies arrangements of elliptic curves in complex abelian surfaces. The first main result of this paper tells us that if $\mathcal{C}$ is an arrangement of elliptic curves in a complex abelian surface $A$, then

$$t_2 + \frac{3}{4}t_4 \geq \sum_{r\geq 5}(2r-9)t_r,$$

so this result is extremely similar to Hirzebruch's inequality for complex line arrangements in the plane (at least in its shape). Using this result Roulleau is able to show that we have

$$h(A;\mathcal{C}) \geq -4.$$

Moreover, the author explains how to construct an infinite series of smooth cubic curve arrangements $C_n$ in $\mathbb{P}^2_{\mathbb{C}}$ admitting **not only ordinary singularities** such that

$$h(\mathbb{P}^2_{\mathbb{C}}; C_n) \rightarrow -4$$

provided that $n$ tends to $\infty$. Based on that observation, we know that $h(\mathbb{P}^2_{\mathbb{C}}) \leq -4$.

The aforementioned construction including smooth cubic curves has a very special geometric origin - it comes from Roulleau-Urzúa's construction of simply connected complex surfaces of general type such that their Chern slopes are dense in $[2,3]$. It is worth mentioning that this construction of surfaces was published in Annals of Mathematics in 2015.

[2] R. Laface, P. Pokora, Local negativity of surfaces with non-negative Kodaira dimension and transversal configurations of curves. *Glasg. Math. J.* **62(1)**: 123–135 (2020).

In this paper we studied Harbourne indices for line arrangements in smooth hypersurfaces of degree $n \geq 3$ in $\mathbb{P}^3_{\mathbb{C}}$. From now on we assume that our line arrangements are connected, i.e., there is no line which does not intersect other lines from the arrangement.

**Theorem 2.21.** *Let $S_n$ be a smooth hypersurface in $\mathbb{P}^3_{\mathbb{C}}$ of degree $n \geq 4$ and let $\mathcal{L}$ be a connected arrangement of $d \geq 2$ lines in $S_n$, then one has*

$$h(\mathcal{L}; S_n) \geq -n(n-1).$$

*Moreover, this bound is sharp and it is achieved by an arrangement consisting of $n$ lines meeting at a single point.*

The second main result of the paper is devoted to transversal arrangements of smooth curves (i.e. arrangements of smooth curves admitting ordinary singularities).

**Theorem 2.22.** *Let $X$ be a smooth complex projective surface with non-negative Kodaira dimension, and let $C = C_1 + ... + C_n \subset X$ be a transversal arrangement of smooth curves having $n \geq 2$ irreducible components. Then*

$$K_X.C + 4\sum_{i=1}^{n}(1 - g(C_i)) - t_2 + \sum_{r\geq 3}(r-4)t_r \leq 3c_2(X) - K_X^2.$$

Using this result we can show that if $H_{\text{elliptic}}(X)$ denotes the global elliptic Harbourne index of a smooth complex projective surface $X$ with non-negative Kodaira dimension (i.e., the infimum is taken over all arrangements of smooth elliptic curves in X with only ordinary singularities), then

$$H_{\text{elliptic}}(X) \geq -4 - (3e(X) - K_X^2),$$

where we denote, as usually, by $e(X)$ the Euler characteristic and by $K_X$ the canonical divisor.

[3] R. Laface, P. Pokora, Towards the weighted bounded negativity conjecture for blowups of algebraic surfaces. *Manuscr. Math.* **163(3-4)**: 361–373 (2020).

Up to that moment, we considered only curve configurations having ordinary singularities, and it is very natural to ask how we can deal with the case of reduced divisors having arbitrary singularities. It seems that for such singularities it is more approachable to study the case of irreducible and reduced curves in surfaces and this is exactly what we are going to do. Our main motivation is the following natural conjecture which is called *the weak bounded negativity conjecture*.

**Conjecture 2.23.** *(WBNC) Let $X$ be a smooth complex projective surface, then there exists an integer $b_w(X)$ such that for all irreducible and reduced curves $C \subset X$ one has*

$$C^2 \geq -b_w(X) \cdot (H.C)^2$$

*for* any *big and nef line bundle $H$ with $H.C > 0$.*

This conjecture has a very interesting consequence, namely if the WBNC is true, then the global Seshadri constant of $X$ at a point $x \in X$ is positive (this is the infimum over single-point Seshadri constants at $x \in X$ with respect to all ample bundles $L \in \mathrm{Pic}(X)$). Our main result here is the following generalization of Orevkov-Zaidenberg's inequality to surfaces with non-negative Kodaira dimension.

**Theorem 2.24.** *Let $X$ be a smooth complex projective surface having non-negative Kodaira dimension. Assume that $C \subset X$ is an irreducible and reduced curve in*

*$X$ having $p_1, ..., p_s$ singular points. We denote by $\mu_i$'s and $m_i$'s the corresponding Milnor numbers and multiplicities of $p_i$'s. Then*

$$\sum_{i=1}^{s}\left(2+\frac{1}{m_i}\right)\mu_i \leq 3e(X) - K_X^2 + 2C^2 + K_X.C.$$

This result allows us to show the following result related to the WBNC.

**Theorem 2.25.** *In the setting of the previous theorem, let $\pi : Y \rightarrow X$ be the blowing up of $X$ along $n$ mutually distinct points. There exists a nef and big line bundle $\Gamma$ on $Y$ such that for every irreducible and reduced curve $C \subset Y$ one has*

$$C^2 \geq -\frac{1}{2}\left(3e(X) - K_X^2\right) - n - \Gamma.C.$$

*In particular, our bound is linear as a $\Gamma$-degree function.*

We can also show the following result.

**Theorem 2.26.** *Let $\pi : X_n \rightarrow \mathbb{P}^2_{\mathbb{C}}$ be the blowing-up at $n$ mutually distinct points. Then for every irreducible and reduced curve $C \subset X_n$ (we can always assume that $C$ is not the exceptional divisor) one has*

$$C^2 \geq -n(C.H),$$

*where $H = \pi^*(\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1))$.*

[4] A. Dimca, B. Harbourne, G. Sticlaru, On the Bounded Negativity Conjecture and singular plane curves. *Mosc. Math. J.* **22(3)**: 427–450 (2022).

In this quite technical paper, the authors develop bounds on numerical characteristics of curves constraining their negativity independently on the characteristic of the underlying field $\mathbb{F}$. The most interesting result, from our very subjective perspective, tells us that the Harbourne index of a rational plane curve $C$ with at most $9$ singular points satisfies $h(\mathbb{P}^2_{\mathbb{F}}, C) > -2$, and this is regardless of the characteristic of $\mathbb{F}$.

2.3. **Clusters of points and constructions of highly singular curves.** Here I will report on $[\mathrm{Hab4}]$ which is a joint work of the applicant and Joaquim Roé. Due to its very technical nature, we present the most important results and we shortly comment on the methods used in the core spots. Here we decided to study the bounded negativity conjecture from a viewpoint of infinitely near points (so speaking more precisely, clusters and weighted clusters of points). Our starting point was a special version of Question 1.3 formulated previously, so let us present it in a convenient form.

**Question 2.27.** *Let $S$ be a smooth complex projective surface, and assume that $b(S) \in \mathbb{Z}$ is a positive integer such that for every irreducible and reduced curve $C \subset S$ one has $C^2 \geq -b(S)$. Let $n \geq 1$ be an integer. Is there a positive integer $b(S, n) \in \mathbb{Z}$ such that for every morphism $S_\pi \xrightarrow{\pi} S$ which is the composition of $n$ point blow-ups, and every irreducible curve $C \subset S_\pi$, one has $C^2 \geq -b(S, n)$?*

If the answer to this question were positive for a given surface $S$ and every $n \geq 1$, then all smooth projective surfaces birational to $S$ would satisfy the BNC, but, as we know, there is not a single surface on which the answer is known for all $n$. Even for the simplest case of $S = \mathbb{P}^2_{\mathbb{C}}$, the existence of $b(\mathbb{P}^2_{\mathbb{C}}, n)$ is unknown for all $n > 9$.

Since the self-intersection of the strict transform $\tilde{C}$ under the blow-up $\pi_p$ of $S$ at a point $p \in C$ is $\tilde{C}^2 = C^2 - \mathrm{m}_p^2(C)$, we expect $b(S, n)$, if it exists, to be an **increasing function** with respect to $n$. In the case of the plane, the existence of irreducible rational nodal curves of every degree, aka Severi curves, shows that if $b(\mathbb{P}^2_{\mathbb{C}}, n)$ exists for all $n$, then

$$\liminf_{n\to\infty}(b(\mathbb{P}^2_{\mathbb{C}}, n)/n) \geq 2.$$

Indeed, a rational nodal curve of degree $d$ has exactly $n = (d-1)(d-2)/2$ nodes, and after blowing up these points its strict transform has self-intersection $d^2 - 4n \simeq -2n + 3\sqrt{2n}$. In particular, $b(\mathbb{P}^2_{\mathbb{C}}, n)$ must grow at least linearly with $n$.

No sequence of irreducible curves with negativity growing faster than $2n$ is known and thus observation has led to the search of new examples considering possibly *reducible reduced curves*. Define

$$h(S, n) = \inf_{\substack{S_\pi \to S \\ n\text{-pt blowup}}} \left\{ \inf_{\substack{C \subset S_\pi \\ \text{reduced}}} \frac{C^2}{n} \right\} \in \mathbb{R} \cup \{-\infty\},$$

so that $b(S, n)$ exists if and only if $h(S, n)$ is finite. An example of arrangements with smooth cubic curves presented by Roulleau show that $\liminf_{n\to\infty} h(\mathbb{P}^2_{\mathbb{C}}, n) \leq -4$, and no sequence of examples has been found with larger than linear growth, so [4, Problem 3.10] asks whether in fact $\lim_{n\to\infty} h(\mathbb{P}^2_{\mathbb{C}}, n) = -4$.

A consequence of our work is that, even if $\inf_n h(\mathbb{P}^2_{\mathbb{C}}, n)$ were finite (which remains unknown), it would not be equal to $C^2/n$ for any curve in a blowup $S_\pi \to \mathbb{P}^2_{\mathbb{C}}$ of $\mathbb{P}^2_{\mathbb{C}}$ at $n$ points.

**Theorem 2.28** ([Hab4]). *Let $h = \inf_n h(\mathbb{P}^2_{\mathbb{C}}, n)$. For every morphism $S_\pi \xrightarrow{\pi} \mathbb{P}^2_{\mathbb{C}}$ which is an $n$ point blow-up, and every reduced curve $C \subset S_\pi$, one has $C^2 > h \cdot n$.*

Now we pass to the discussion on Harbourne indices for reduced curves on smooth projective surfaces. As we observed in Introduction, these indices can be viewed as the average intersection numbers of negative curves by the number of singular points that they possess. A general definition that is introducing the notion of a Harbourne constant, especially with respect to clusters of points, has the following form.

**Definition 2.29.** *Let $C \subset \mathbb{P}^2$ be a reduced curve of degree $\mathrm{d}$, and let $K \subset \mathbb{P}^2_{\mathbb{C}}$ be a finite set. The Harbourne constant of $C$ at $K$ is defined as*

$$H(C, K) = \frac{d^2 - \sum_{p \in K} \mathrm{m}_p(C)^2}{|K|},$$

*where $|K|$ denotes the cardinality of $K$. It is often useful to generalize this notion and allow for infinitely near points. In other words, one assumes that $p_i \in S_i$ where $S_1 = \mathbb{P}^2_{\mathbb{C}}$ and $\pi_i : S_{i+1} \to S_i$ is the blowup centered at $p_i$, and then $\mathrm{m}_{p_i}(C)$ is replaced by the multiplicity of the strict transform of $C$ at $p_i$.*

*In this setting, the Harbourne index of a curve $C$ with ordinary singularities is the Harbourne constant of $C$ at the set of singular points, namely*

$$h(\mathbb{P}^2_{\mathbb{C}}; C) = H(C, \mathrm{Sing}(C)).$$

Up to that moment, the most negative Harbourne index for curves with ordinary singularities found in the literature is given by Wiman's arrangement of lines $\mathcal{W}$, which has $h(\mathbb{P}^2_{\mathbb{C}}; \mathcal{W}) = -225/67 \simeq -3.358$. In $[\mathrm{Hab4}]$, we constructed the most negative known example of a reduced plane curve with only ordinary singularities.

**Theorem 2.30** ([Hab4])**.** *There exist reduced curves $C \subset \mathbb{P}^2_{\mathbb{C}}$ with ordinary singularities and Harbourne indices $h(\mathbb{P}^2_{\mathbb{C}}; C)$ arbitrarily close to $-25/7 \simeq -3.571$.*

In order to construct such curves, we apply the classical Kummer covering

$$\pi : \mathbb{P}^2_{\mathbb{C}} \ni [x : y : z] \rightarrow [x^k : y^k : z^k] \in \mathbb{P}^2_{\mathbb{C}}$$

branched along $xyz = 0$ with $k \in \mathbb{Z}_{\geq 2}$ to Wiman's arrangement of $45$ lines. Taking consecutive large values of $k$, we can observe that the sequence of curves $C_k$ that we obtained by taking the preimages with respect to $\pi$ has the following property: the Harbourne index $h(\mathbb{P}^2_{\mathbb{C}}; C_k)$ tends to the value $-\frac{225}{67} \cdot \frac{201}{198} = -\frac{25}{7} \approx -3.571$ provided that $k \rightarrow \infty$.

### Further developments regarding Harbourne indices, clusters of points, and constructions of highly singular curves via ramified morphisms

[1] P. Pokora and J. Roé, The 21 reducible polars of Klein's quartic. *Exp. Math.* **30(1)**: 1 – 18 (2021).

In this paper we construct a family of the so-called Klein's arrangements of curves. These curves are constructed via the gradient map given by the partials of the standardized equation of the Klein quartic curve

$$\Phi_4 \, : \, xy^3 + yz^3 + zx^3 = 0.$$

It is well-known that the Klein quartic has the largest possible order of the automorphism group (according to Hurwitz's bound) among curves of degree $4$, namely $168$. This group acts on the complex projective plane. In particular, using this group (or its extension to $\mathbb{C}^3$ which is the Shephard-Todd irreducible reflection group $G_{24}$) we can construct an arrangement of $21$ complex lines having $21$ quadruple and $28$ triple intersection points - this the well-known Klein's arrangement of lines. Using the gradient map and a suitable chosen defining equation of the Klein arrangement of lines, we get a curve of degree $63$ which splits into two orbits, one consists of $21$ lines, and the second consists of $21$ smooth conics. Each line intersects exactly one conic in two points, so we obtain $2 \cdot 21 = 42$ double, $9 \cdot 21 = 189$ quadruple and $9 \cdot 28 = 252$ triple intersection points. It turns out that we can keep continuing successfully this procedure, for instance we can take the preimage of the curve of degree $63$ obtaining an arrangement of curves consisting of $21$ lines, $21$ conics, and $21$ curves of degree $6$ (each type of curves forms an orbit of the length $21$). However, in this case we know that singularities are no longer ordinary, and we are still missing a complete picture of all singular points.

[2] I. Dolgachev, A. Laface, U. Persson, G. Urzúa, Chilean configuration of conics, lines and points. *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)* **23(2)**: 877–914 (2022).

In this extremely interesting and insightful paper the authors construct a highly non-trivial arrangement of $12$ smooth conics having exactly $9$ eightfold points and $12$ double points that can be considered as a natural generalization of the famous Hesse arrangement of $12$ lines in the complex projective plane. The arrangement of $12$ conics is right now called Chilean arrangement of $12$ conics and it realizes an abstract configuration of conics and points $(12_6, 9_8)$. This arrangement is constructed with the use of an Halphen pencil of index $2$ which contains exactly four reducible members, each is the union of three smooth conics. Moreover, the authors show that any Halphen elliptic fibration of index $2$ with four singular fibers arises from such a configuration of conics which stands as the uniqueness result for the Chilean arrangement.

[3] C. Galindo, F. Monserrat, C.-J. Moreno-Ávila, E. Pérez-Callejo, On the Degree of Curves with Prescribed Multiplicities and Bounded Negativity. *Int. Math. Res. Not.*, `https://doi.org/10.1093/imrn/rnac085`.

In this paper the authors provide a lower bound on the degree of curves of the projective plane $\mathbb{P}^2_{\mathbb{C}}$ passing through the centers of a divisorial valuation $\nu$ of $\mathbb{P}^2_{\mathbb{C}}$ with prescribed multiplicities. They also give some results related to the bounded negativity conjecture concerning those rational surfaces having the projective plane as a relatively minimal model.

2.4. **Conic-line arrangements in the complex projective plane and their freeness, negativity and combinatorics.** Our main aim for this section is to present techniques and tools developed by the applicant with co-authors towards better understanding of the combinatorics and geometry of rational curve arrangements. Undoubtedly the theory of line arrangements is a classical and rich subject of studies with many deep results with applications and impact in numerous branches of mathematics. However, if we look at the theory of curve arrangements in the plane, to our surprise, a lot of work has to be done in order to reach the same level of understanding. This is the main reason why we decided to start working systematically with arrangements of smooth conics and lines in the complex projective plane. For the first part of our discussion, let us present our setting. We are going to consider arrangements of smooth rational curves (smooth conics and lines) having only ordinary singularities. This assumption might be considered as a strong restriction, but the most important advantage of this approach is that we can apply some combinatorial methods which are hardly applicable if we start to work in the whole generality.

If $\mathcal{CL} = \{\ell_1, ..., \ell_d, C_1, ..., C_k\} \subset \mathbb{P}^2_{\mathbb{C}}$ is an arrangement of $d$ lines and $k$ conics having only ordinary singularities, i.e., the intersection points look locally as $\{x^k = y^k\}$ for some integer $k \geq 2$, then we have the following combinatorial count (by Bézout):

$$4\binom{k}{2} + \binom{d}{2} + 2kd = \sum_{r\geq 2}\binom{r}{2}t_r,$$

where $t_r$ denotes the number of $r$-fold points, i.e., points where exactly $r$ curves from $\mathcal{CL}$ meet.

First of all, we showed a de Bruijn-Erdős type result which provides a lower bound on the number of intersection points for a certain class of conic-line arrangements - this result is directly inspired by combinatorial considerations presented and studied in [Hab5].

**Theorem 2.31** ([Hab7])**.** *Let $\mathcal{CL} = \{\ell_1, ..., \ell_d, C_1, ..., C_k\} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $d \geq 2$ lines and $k \geq 2$ conics having only ordinary singularities as the intersections. Furthermore, assume that $t_{k+d} = t_{k+d-1} = t_{k+d-2} = t_{k+d-3} = 0$, then one always has*

$$f_0 = \sum_{r \geq 2} t_r \geq k + d.$$

The proof of this statement is a merger of classical geometric considerations and properties of the intersection form that are decoded by the Hodge index theorem.

In the next step, we focused on a Hirzebruch-type inequality for conic-line arrangement and we provided the following results that can be considered as the main contribution from the paper [Hab7].

**Theorem 2.32.** *Let $\mathcal{CL} = \{\ell_1, ..., \ell_d, C_1, ..., C_k\} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $d \geq 6$ lines and $k \geq 2$ conics such that all intersection points are ordinary singularities. Moreover, we assume that $t_{d+k} = 0$ and one can can find a subarrangement of $\mathcal{CL}$ consisting of $6$ lines intersecting only along double and triple points. Then one has*

$$8k + t_2 + \frac{3}{4}t_3 \geq d + \sum_{r \geq 5}(2r - 9)t_r.$$

In order to show the above inequality, we used the theory of abelian cover proposed by Namba - this construction is more involved comparing with Hirzebruch's papers and there is not so much flexibility in considerations.

Now we are ready to present certain applications. Our first result is devoted to the local negativity problem.

**Theorem 2.33.** *Let $\mathcal{CL} \subset \mathbb{P}^2_{\mathbb{C}}$ be a conic-line arrangement satisfying the assumptions of Theorem 2.32, then one has*

$$h(\mathbb{P}^2_{\mathbb{C}}; \mathcal{CL}) \geq -4.5.$$

Next, we studied log-surfaces and their Chern slopes. Let $\mathcal{CL} = \{\ell_1, ..., \ell_d, C_1, ..., C_k\} \subseteq \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $k$ conics and $d$ lines having only ordinary singularities. Assume additionally that $t_{k+d} = 0$. Consider the blowing up $f : X \rightarrow \mathbb{P}^2_{\mathbb{C}}$ along the set of singular points of $\mathcal{CL}$ having multiplicity $\geq 3$. Denote by $\overline{\mathcal{CL}}$ the reduced total transform of $\mathcal{CL}$. Then the pair $(X, \overline{\mathcal{CL}})$ is a log-surface. We can easily compute the Chern numbers of pair $(X, \overline{\mathcal{CL}})$, namely

$$\overline{c}_1^2(X, \overline{\mathcal{CL}}) = 9 - 5d - 8k + \sum_{r \geq 2}(3r - 4)t_r;$$

$$\overline{c}_2(X, \overline{\mathcal{CL}}) = 3 - 2d - 2k + \sum_{r \geq 2}(r - 1)t_r.$$

Let us recall that by a result due to Miyaoka and Sakai, if the logarithmic Kodaira dimension of pair $(X, \overline{\mathcal{CL}})$ is equal to 2, i.e., $\overline{\kappa}(X, \overline{\mathcal{CL}}) = 2$, then we have

$$\overline{c}_1^2(X, \overline{\mathcal{CL}}) \leq 3\overline{c}_2(X, \overline{\mathcal{CL}}).$$

If we restrict our attention to the case of lines, i.e., $k = 0$, then by a result due to Sommese [33, Theorem 5.1] we know that one always has

$$\overline{c}_1^2(X, \overline{\mathcal{L}}) \leq \frac{8}{3}\overline{c}_2(X, \overline{\mathcal{L}}),$$

and the equality holds if and only if $\mathcal{L}$ is projectively equivalent to the dual Hesse arrangement of lines $\mathcal{H}$.
If the ground field $\mathbb{F}$ is arbitrary, then Eterović, Figueroa, and Urzúa in [14] proved that

$$\frac{2d-6}{d-2} \leq \frac{\overline{c}_1^2(X, \overline{\mathcal{L}})}{\overline{c}_2(X, \overline{\mathcal{L}})} \leq 3,$$

and the left-hand side equality holds if and only if $\mathcal{L}$ is a star configuration, i.e., it has only double intersection points, and the right hand side equality holds if and only if $\sum_{r\geq 2} t_r = d$, so this is the case, for instance, when $\mathcal{L}$ is a finite projective plane arrangement. Among others, they also found an interesting link between Harbourne indices of line arrangements and the limit points of ratios of log-Chern numbers – in fact one can show that the accumulation points of $H$-indices of line arrangements $\mathcal{L}$ are in one-to-one correspondence with the limit points of $\frac{\overline{c}_1^2(X,\overline{\mathcal{L}})}{\overline{c}_2(X,\overline{\mathcal{L}})}$, see [14, Proposition 4.9].

If we focus on the case of conic arrangements, i.e., $d = 0$, then the applicant in [27] proved that one always has

$$\overline{c}_1^2(X, \overline{\mathcal{C}}) < \frac{8}{3}\overline{c}_2(X, \overline{\mathcal{C}}),$$

but we do not have any example of a conic arrangement $\mathcal{C}$ such that for the associated pair $(X, \overline{\mathcal{C}})$ one has $\frac{\overline{c}_1^2(X,\overline{\mathcal{C}})}{\overline{c}_2(X,\overline{\mathcal{C}})} \approx \frac{8}{3}$. Now we are going to study some extremal conic-line arrangements from the viewpoint of log-Chern slopes. Let us call the ratio

$$E(X, \overline{\mathcal{CL}}) := \frac{\overline{c}_1^2(X, \overline{\mathcal{CL}})}{\overline{c}_2(X, \overline{\mathcal{CL}})}$$

is the log-Chern slope of $(X, \overline{\mathcal{CL}})$. Before we present our main contribution, we want to recall the following (modified) question by Urzúa [37, Question VII.12] which is a main motivation for our studies.

**Question 2.34.** *Let $\mathcal{CL} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $k \geq 2$ conics and $d \geq 2$ lines such that all intersection points are ordinary. Is it true that $E(X, \overline{\mathcal{CL}}) \leq \frac{8}{3}$?*

Unfortunately, we do not know whether the answer is yes, but we were able to construct conic-line arrangements having high values of $E(X, \overline{\mathcal{CL}})$. Our record comes from a construction that was established in the aforementioned paper on Klein's arrangements written by the applicant and Roé.

*Example* 6. Let us recall that Klein's arrangement of conics and lines in the complex projective plane is an arrangement consisting of $21$ lines and $21$ conics (these curves are polars to Klein's quartic curve at the $21$ quadruple points of Klein's arrangement of $21$ lines), and it has $42$ double points, $252$ triple points, and $189$ quadruple points. Simple computations give

$$E(X, \overline{\mathcal{CL}}) = \frac{\overline{c}_1^2(X, \overline{\mathcal{CL}})}{\overline{c}_2(X, \overline{\mathcal{CL}})} = \frac{9 - 8\cdot 21 - 5\cdot 21 + 2\cdot 42 + 5\cdot 252 + 8\cdot 189}{3 - 2\cdot 21 - 2\cdot 21 + 42 + 2\cdot 252 + 3\cdot 189} \approx 2.512,$$

and this is the highest known value of log-Chern slopes in the class of conic-line arrangements.

Now we are passing to the questions revolving around freeness of conic-line arrangements in the complex projective plane, this is a part of a joint project with Alexandru Dimca which consists of several papers (right now we have 2 published articles, 2 preprints, and at least one forthcoming note).

We start with presenting our set-up. Let $\mathcal{CL} = \{\ell_1, ..., \ell_d, C_1, ..., C_k\} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement consisting of $d$ lines and $k$ smooth conics. We assume that our conic-line arrangements have $n_2$ nodes, $t$ tacnodes, and $n_3$ ordinary triple points. We have the following combinatorial count

$$4\binom{k}{2} + \binom{d}{2} + 2kd = n_2 + 2t + 3n_3.$$

The first result for this part of the achievement is the following Hirzebruch-type inequality that comes from $[\mathrm{Hab6}]$.

**Theorem 2.35.** *Let $\mathcal{CL} = \{\ell_1, ..., \ell_d, C_1, ..., C_k\} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $d$ lines and $k$ smooth conics and such that $2k + d \geq 12$. Assume that $\mathcal{CL}$ has only $n_2$ nodes, $t$ tacnodes, and $n_3$ ordinary triple points. Then*

$$20k + n_2 + \frac{3}{4}n_3 \geq d + 4t.$$

The proof is based on an orbifold version of the Bogomolov-Miyaoka-Yau inequality presented by Langer in [22]. Then, using the spectra of singularities and techniques developed by Varchenko and Steenbrink, we showed the following result that also comes from $[\mathrm{Hab6}]$.

**Theorem 2.36.** *Let $\mathcal{CL} = \{\ell_1, ..., \ell_d, C_1, ..., C_k\} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $d \geq 0$ lines and $k \geq 0$ smooth conics. Assume that $\mathcal{CL}$ has only $n_2$ nodes, $t$ tacnodes, and $n_3$ ordinary triple points. Let $C = \ell_1 + ... + \ell_d + C_1 + ... + C_k$ and write $m := \deg C = d + 2k$ as $m = 3m' + \epsilon$ with $\epsilon \in \{1, 2, 3\}$. Then one has*

$$t + n_3 \leq \binom{m-1}{2} + k - \frac{m'(5m'-3)}{2}$$

*and*

$$n_3 \leq (m'+1)(2m'+1).$$

The main contribution in the field of the freeness of conic-line arrangements with nodes, tacnodes, and ordinary triple points is the following complete classification of such arrangements $[\mathrm{Hab6}]$.

**Theorem 2.37.** *Let $\mathcal{CL}$ be an arrangement of $d \geq 1$ lines and $k \geq 1$ smooth conics having only nodes, tacnodes, and ordinary triple points as singularities. Then $\mathcal{CL}$ is free if and only if one of the following cases occur. In each case we list the numbers $n_2$, $t$, and $n_3$ of nodes, tacnodes, and ordinary triple points, respectively.*

*(1) $d = k = 1$ and $\mathcal{CL}$ consists of a smooth conic and a tangent line. In this case, $n_2 = n_3 = 0$, $t = 1$.*
*(2) $d = 2$, $k = 1$ and $\mathcal{CL}$ consists of a smooth conic and two tangent lines. In this case $n_2 = 1$, $n_3 = 0$, $t = 2$.*
*(3) $d = 3$, $k = 1$ and either $\mathcal{CL}$ is a smooth conic inscribed in a triangle, or $\mathcal{CL}$ is a smooth conic circumscribed in a triangle. In the first case we have $n_2 = 3$, $n_3 = 0$, $t = 3$, and in the second case we have $n_2 = t = 0$, $n_3 = 3$.*
*(4) $d = 3$, $k = 2$ and $\mathcal{CL}$ consists of a triangle $\Delta$, a smooth conic inscribed in $\Delta$, and another smooth conic circumscribed in $\Delta$. In this case, $n_2 = 0$, $n_3 = 3$, $t = 5$.*

*In particular, a free conic-line arrangement having only nodes, tacnodes, and ordinary triple points is determined up to a projective equivalence by the numerical data $n_2$, $n_3$ and $t$.*

As a simple corollary, we obtain the following.

**Corollary 2.38.** *Numerical Terao's Conjecture holds for conic-line arrangements with nodes, tacnodes, and ordinary triple points.*

### Further developments regarding the freeness and nearly-freeness of reduced plane curves

Here we present briefly very selected results revolved around the freeness of reduced plane curves in the complex projective plane.

[1] A. Dimca, M. Janasz, P. Pokora, On plane conic arrangements with nodes and tacnodes. *Innov. Incidence Geom.* **19(2)**: 47 – 58 (2022).

In that paper we studied arrangements of smooth conics having only nodes and tacnodes as singularities. We showed that if $\mathcal{C} = \{C_1, ..., C_k\} \subset \mathbb{P}^2_{\mathbb{C}}$ is an arrangement of $k \geq 6$ smooth conics which admits only nodes and tacnodes as singularities, then

$$t \leq \frac{k^2}{3} + 3k.$$

In particular, this improves Miyaoka's bound on the number of tacnodes presented in [25]. Moreover, we provide a complete characterization of nearly-free conic arrangements with nodes and tacnodes.

**Theorem 2.39.** *Let $\mathcal{C} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $k \geq 2$ smooth conics with only nodes and tacnodes as singularities. Then $\mathcal{C}$ is nearly-free if and only if*

$$k \leq 4 \text{ and } t = k(k-1).$$

[2] P. Pokora, $\mathcal{Q}$-conic arrangements in the complex projective plane. Electronically available at **arXiv:2203.11503**.

The main aim of the note was to study the freeness of smooth conic arrangements with nodes, tacnodes, and ordinary tripe and quadruple points. The main result of the note gives us the following.

**Theorem 2.40.** *There does not exist any arrangement with $k \geq 3$ conics in the complex projective plane with nodes, tacnodes, ordinary triple and quadruple points which is free.*

[3] A. Gałecka, On the nearly freeness of conic-line arrangements with nodes, tacnodes, and ordinary triple points. *Bol. Soc. Mat. Mex., III. Ser.* **28(3)**: Paper No. 67, 12 p. (2022).

The author shows that if there exists a nearly-free conic-line arrangement in the complex plane with nodes, tacnodes, and ordinary triple points, then its degree is less or equal to 12. In the next step, the author provides examples of such conic-line arrangements having degree $d \in \{3, 4, 5, 6, 7\}$. Then, based on a Hirzebruch-type inequality for such conic-line arrangements, the author is able to show that there are no conic-line arrangements with nodes, tacnodes, and ordinary triple points if degree $d \in \{10, 11, 12\}$. Based on that discussion, it remains to decide whether we can find examples of nearly-free conic-line arrangements in degree $8$ and $9$, but this problem seems to be very difficult.

[4] A. Dimca, P. Pokora, Maximizing curves viewed as free curves. Electronically available at **arXiv:2208.13399**.

The main aim of the paper is to establish a direct link between reduced free complex plane curves with ADE-singularities and the so-called maximizing curves, i.e., reduced curves $C \subset \mathbb{P}^2_{\mathbb{C}}$ with ADE-singularities of even degree $n \geq 4$ such that

$$\tau(C) = 3\frac{n}{2}\left(\frac{n}{2} - 1\right) + 1,$$

where $\tau(C)$ denotes the total Tjurina number of $C$. Our main result can be formulated as follows.

**Theorem 2.41.** *Let $C$ be a plane curve of degree $n = 2m \geq 4$ having only* ADE-*singularities. Then $C$ is maximizing if and only if $C$ is a free curve with the exponents $(m-1, m)$.*

2.5. **Hirzebruch-type inequality and their applications towards extreme combinatorial problems.** This part, aimed to be less technical than the previous section, is to report on $[\mathrm{Hab5}]$. This article combines features of a modern survey on Hirzebruch-type inequalities and applications towards better understanding of extreme point-line configurations in the plane and a research article (since it contains new results). The main purpose of this article was to provide an introduction, algebro-topological in its nature, to Hirzebruch-type inequalities for plane curve arrangements in the complex projective plane and our goal was to present a summary of the technicalities and some recent combinatorial applications, for instance in the context of the Weak Dirac Conjecture. Moreover, we put into one spot relevant results obtained by the applicant in the framework of Hirzebruch-type inequalities for reduced plane curves in [27, 28] and a combinatorial approach towards constructing new examples of ball-quotient surfaces [6, 27, 29]. The most important part is devoted to an understandable explanation (on the level of combinatorics and algebra) of Hirzebruch's proof of his famous inequality for complex line arrangements. We present also other inequalities for line arrangements obtained with use of an orbifold version of the Bogomolov-Miyaoka-Yau inequalities. Let us recall results obtained by the applicant in [28] that are discussed in $[\mathrm{Hab5}]$.

**Theorem 2.42.** *Let $\mathcal{C} = \{C_1, ..., C_\tau\} \subset \mathbb{P}^2_{\mathbb{C}}$ be a $d$-arrangement of $\tau \geq 3$ curves with $d \geq 2$. Then*

$$t_2 + \frac{3}{4}t_3 + d^2\tau(d\tau - \tau - 1) \geq \sum_{r\geq 5}\left(\frac{r^2}{4} - r\right)t_r.$$

**Theorem 2.43.** *Let $\mathcal{CL} = \{\ell_1, ..., \ell_d, C_1, ..., C_k\}$ be an arrangement of $d$ lines and $k$ conics such that $t_r = 0$ for $r > \frac{2(d+2k)}{3}$, and we assume that all intersection points of the arrangement are ordinary singularities. Then*

$$t_2 + \frac{3}{4}t_3 + (4k + 2d - 4)k \geq d + \sum_{r \geq 5} \left(\frac{r^2}{4} - r\right) t_r.$$

In the second part of the paper (which has a purely survey role), where algebraic geometry meets combinatorics, we recall the most important results in the extreme theory of point-line configurations in the plane obtained by using Hirzebruch-type inequalities. Let us denote by $\mathcal{P} \subset \mathbb{P}^2_{\mathbb{C}}$ a finite set of $n$ mutually distinct points and let $\mathcal{L}(\mathcal{P})$ be the set of lines determined by $\mathcal{P}$, where a line that passes through at least two points from $\mathcal{P}$ is said to be determined by $\mathcal{P}$.

As a starting point for our discussion we recall the original Dirac conjecture.

**Conjecture 2.44** (Dirac)**.** *Every set $\mathcal{P}$ of $n$ non-collinear points contains a point in at least $\frac{n}{2}$ lines determined by $\mathcal{P}$.*

It turned out that the Dirac conjecture is false – the smallest counterexample has $n = 7$ points, namely the vertices of a triangle together with the midpoints of its sides and its centroid. However, the conjecture was resolved positively by Green and Tao in [16] for very large $n$. In this view, we can formulate the actual Dirac conjecture which is, according to our best knowledge, open.

**Conjecture 2.45.** *There is a constant $c$ such that every set $\mathcal{P}$ of $n$ non-collinear points contains a point in at least $\frac{n}{2} - c$ lines determined by $\mathcal{P}$.*

In 1961, P. Erdős proposed the following Weak Dirac Conjecture [13].

**Conjecture 2.46** (WDC)**.** *Every set $\mathcal{P}$ of $n$ non-collinear points in the plane (presumably over the real numbers) contains a point which is incident to at least $\lceil \frac{n}{c} \rceil$ lines from $\mathcal{L}(\mathcal{P})$ for some constant $c > 0$.*

The Weak Dirac Conjecture was proved independently by Beck [5] and Szemerédi-Trotter [34], but they did not specify the actual value of $c$. It is worth noticing that in [21, Chapter 6], the authors explained that it is plausible to believe that $c = 3$, and it turned out that this prediction is correct [17].

**Theorem 2.47** (Han)**.** *The Weak Dirac Conjecture holds with $c = 3$.*

The proof, very simple in its nature, is directly based on Bojanowski-Langer's inequality [7].

**Theorem 2.48.** *Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be a line arrangement with $\tau \geq 6$ such that $t_r = 0$ for $r > \frac{2\tau}{3}$. Then we have*

$$t_2 + \frac{3}{4}t_3 \geq \tau + \sum_{r \geq 5} \left(\frac{r^2}{4} - r\right) t_r.$$

Finally, let us comment on another important result in the extreme theory of point-line configurations in the plane, namely Beck's theorem [5].

**Theorem 2.49** (Beck)**.** *For a finite set $\mathcal{P}$ of $n$ points in $\mathbb{R}^2$ one of the following is true:*

- *there exists a line that contains $c_1 n$ points from $\mathcal{P}$ for some positive $c_1$;*

- *there are at least $c_2 n^2$ lines determined by $\mathcal{P}$.*

Using Langer's results revolved around Hirzebruch's type inequalities, de Zeeuw in [9] prove the following result that provides the strongest known version of Beck's theorem.

**Theorem 2.50** (de Zeeuw)**.** *Let $\mathcal{P}$ be a finite set of $n$ points in $\mathbb{R}^2$, then one of the following is true:*

- *there is a line that contains more than $\frac{6+\sqrt{3}}{9}n$ points of $\mathcal{P}$;*
- *there are at least $\frac{n^2}{9}$ lines determined by $\mathcal{P}$.*

## 3. Summary

My research presented as the habilitation achievement is carried out within the theory of curves in algebraic surfaces, but it reaches out to other areas due to the interdisciplinary nature of the objects that I am studying. The main subjects of my studies are variations on Terao's freeness conjecture which combines both combinatorial and algebraic methods, and the bounded negativity conjecture, one of the oldest and the most fundamental open problems in the theory of algebraic surfaces. My habilitation achievement gives a substantial progress towards better understanding of singular curves in the context of their local negativity and they provide a bridge between negative curves and the Zariski decompositions. Moreover, results towards Numerical Terao's freeness conjecture explain also the non-existence of certain presumably highly singular curves. In order to prove many results devoted to the bounded negativity conjecture, I obtained several Hirzebruch-type inequalities which have a very combinatorial nature. Due to this reason, my results are of interests of combinatorialists working on the extreme point-curve problems. Moreover, constructing singular curves is very important due to its applications towards the containment problems, i.e., questions about the containment of symbolic powers of homogeneous ideals associated with sets of points and their algebraic powers. This makes my habilitation achievement even stronger due to possible applications in different subfields of study.

## References


[1] M. Barakat and L. Kühne, Computing the nonfree locus of the moduli space of arrangements and Terao's freeness conjecture. **arXiv:2112.13065**, to appear in *Mathematics of Computation*.

[2] Th. Bauer, Cr. Bocci, S. Cooper, S. di Rocco, M. Dumnicki, B. Harbourne, K. Jabbush, A. L. Knutsen, A. Küronya, R. Miranda, J. Roé, H. Schenck, T. Szemberg, Z. Teiler, Recent developments and open problems in linear series. In *Contributions to Algebraic Geometry*, 93–140, IMPANGA Lecture Notes (Piotr Pragacz, ed.), EMS Series of Congress Reports, edited by the European Mathematical Society Publishing House 2012.

[3] Th. Bauer, B. Harbourne, A. L. Knutsen, A. Küronya, S. Müller–Stach, X. Roulleau, T. Szemberg, Negative curves on algebraic surfaces. *Duke Math. J.* **162**: 1877 – 1894 (2013).

[4] Th. Bauer, S. Di Rocco, B. Harbourne, J. Huizenga, A. Lundman, P. Pokora, T. Szemberg, Bounded Negativity and Arrangements of Lines. *International Mathematical Research Notices* **vol. 2015**: 9456 - 9471 (2015).

[5] J. Beck, On the lattice property of the plane and some problems of Dirac, Motzkin and Erdős in combinatorial geometry. *Combinatorica*, **3(3-4)**: 281 – 297 (1983).

[6] J. Bokowski and P. Pokora, On line and pseudoline configurations and ball-quotients. *Ars Mathematica Contemporanea* **13(2)**: 409 – 416 (2017).

[7] R. Bojanowski, *Zastosowania uogólnionej nierówności Bogomolova-Miyaoka-Yau*. Master Thesis (in Polish), `http://www.mimuw.edu.pl/%7Ealan/postscript/bojanowski.ps`, 2003.

[8] R. Cheng and R. van Dobben de Bruyn, Unbounded negativity on rational surfaces in positive characteristic. *J. Reine Angew. Math.* **783**: 217-226 (2022).

[9] F. de Zeeuw, Spanned lines and Langer's inequality. **arXiv:1802.08015**.

[10] A. Dimca, *Hyperplane arrangements. An introduction*. Universitext Cham: Springer. xii, 200 p. (2017).

[11] A. Dimca and G. Sticlaru, Free and Nearly Free Curves vs. Rational Cuspidal Plane Curves. *Publ. Res. Inst. Math. Sci.* **54(1)**: 163 – 179 (2018).

[12] S. Di Rocco, A. Küronya, S. Müller-Stach, T. Szemberg, *Mini-Workshop: Negative Curves on Algebraic Surfaces*. Oberwolfach Reports `doi:10.4171/OWR/2014/10`: 555 - 594 (2014).

[13] P. Erdős, Some unsolved problems. *Magyar Tud. Akad. Mat. Kutato Int. Közl* **6**: 221 – 254 (1961).

[14] S. Eterović, F. Figueroa, G. Urzúa, On the geography of line arrangements. *Advances in Geometry* **22(2)**: 269 – 276 (2022).

[15] T. Fujita, On Zariski problem. *Proc. Japan Acad.* **55, Ser. A**: 106 – 110 (1979).

[16] B. Green and T. Tao, On sets defining few ordinary lines. *Discrete Comput. Geom.* **50(2)**: 409 – 468 (2013).

[17] Z. Han, A note on the weak Dirac conjecture. *The Electronic Journal of Combinatorics* **24**:#63 (2017), `http://www.combinatorics.org/v24i1p63`.

[18] B. Harbourne, Global aspects of the geometry of surfaces. *Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica* **vol. IX**: 5 – 41 (2010).

[19] F. Hirzebruch, Arrangement of lines and Algebraic surfaces. Arithmetic and geometry, Vol. II, 113 – 140, Progr. Math., 36, Birkhäuser, Boston, Mass., 1983.

[20] F. Hirzebruch, Singularities of algebraic surfaces and characteristic numbers. *The Lefschetz centennial conference, Part I (Mexico City, 1984) Contemp. Math.* **58**: 141 – 155 (1986).

[21] V. Klee and S. Wagon, *Old and New Unsolved Problems in Plane Geometry and Number Theory*. The Mathematical Association of America, 1991.

[22] A. Langer, Logarithmic orbifold Euler numbers with applications. *Proc. London Math. Soc.* **(3) 86**: 358 – 396 (2003).

[23] R. Lazarsfeld, *Positivity in Algebraic Geometry, I & II Ergebnisse der Mathematik und ihrer Grenzgebiete*, Vols. 48 & 49, Springer Verlag, Berlin, 2004.

[24] Y. Miyaoka, On the Chern numbers of surfaces of general type. *Invent. Math.* **42(1)**: 225 – 237 (1977).

[25] Y. Miyaoka, The maximal number of quotient singularities on surfaces with given numerical invariants. *Math. Ann.* **268**: 159 – 171 (1984).

[26] P. Orlik, H. Terao, *Arrangements of Hyperplanes*. Grundlehren der Mathematischen Wissenschaften 300. Springer-Verlag, Berlin, 1992.

[27] P. Pokora, Hirzebruch type inequalities and plane curve configurations. *International Journal of Mathematics* **28(2)**: 1750013 (11 pages) (2017).

[28] P. Pokora, The orbifold Langer-Miyaoka-Yau inequality and Hirzebruch-type inequalities. *Electronic Research Announcements in Mathematical Sciences* **24**: 21 – 27 (2017).

[29] P. Pokora, Hirzebruch-Kummer covers of algebraic surfaces. *Turk. J. Math.* **43(1)**: 412 – 421 (2019).

[30] P. Pokora and H. Tutaj-Gasińska, Harbourne constants and conic configurations on the projective plane. *Math. Nachr.* **289(7)**: 888-894 (2016).

[31] F. Severi, *Vorlesungen über algebraische Geometrie*. Teubner (1921).

[32] H. Schenck and S. Tohaneanu, Freeness of conic-line arrangements in $\mathbb{P}^2$. *Comment. Math. Helv.* **84(2)**: 235 – 258 (2009).

[33] A. J. Sommese, On the density of ratios of Chern classes of algebraic surfaces. *Math. Annalen* **268**: 207 – 221 (1984).

[34] E. Szemerédi and W. T. Trotter, Extremal problems in discrete geometry. *Combinatorica*, **3(3-4)**: 381 – 392 (1983).

[35] D. Testa, T. Várilly-Alvarado, M. Velasco, Big rational surfaces. *Math. Ann.* **351(1)**: 95 – 107 (2011).

[36] H. Terao, Generalized Exponents of a Free Arrangement of Hyperplanes and Shepherd-Todd-Brieskorn Formula. *Invent. Math.* **63**: 159 – 179 (1981).

[37] G. A. Urzúa, *Arrangements of curves and algebraic surfaces*. Thesis (Ph.D.) University of Michigan, 166 pages, ISBN: 978-0549-82049-9, 2008.

[38] O. Zariski, The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface. *Ann. Math.* **76**: 560 – 615 (1962).

[39] A. Wiman, Zur Theorie der endlichen Gruppen von birationalen Transformationen in der Ebene. *Math. Annalen* **48**: 195 – 240 (1896).

## Presentation of significant scientific activity carried out at more than one university, scientific or cultural institution, especially at foreign institutions

I began my scientific activity carried out outside my University, during the PhD Programme at the Pedagogical University of Cracow, in Poland and in Germany almost simultaneously. In 2013, I spent one semester in the Warsaw Centre for Mathematics and Computer Sciences where I was able to take part in an advanced course in algebraic geometry devoted to the positivity in algebraic geometry. At that time I started my international collaboration with David Schmitz (Marburg). Shortly afterwards I spent my first short internship in Bonn (January 2014) and I had a great chance to discuss mathematics with Robert Lazarsfeld, a world-renowned expert in positivity in algebraic geometry. In 2014, I spend one whole semester at Albert-Ludwigs Univerität Freiburg im Breisgau (in the framework of ERASMUS+ Exchange Programme) working in the group run by Prof. Stefan Kebekus, and then the whole winter semester of the academic year 2014/2015 at Philipps Universität Marburg (an der Lahn) working in the research group run by Prof. Thomas Bauer - my stay there was supported by the DAAD Scholarship awarded in June 2014. Shortly after defending my PhD thesis, I applied for a Post-Doc position at Johannes Gutenberg Universität in Mainz, and my application was accepted by the selection committee and then I was attached to the research group run by Prof. Manfred Lehn. In that period, I started many research projects in the international collaborations, mostly working with Xavier Roulleau (Professor at Aix Marseilles), Jürgen Bokowski (Professor Emeritus in Darmstadt), Roberto Laface (PhD student and then Post-Doc in Hannover), Joaquim Roé (Professor at Autonoma Barcelona). After one year position in Mainz, I spent one additional year in Hannover working in the research group run by Prof. Klaus Hulek. During that time I continued my international collaboration and I obtained the academic title *Privatdozent*. In November 2017, I came back to Poland and from that time I worked at the Institute of Mathematics Polish Academy of Sciences and from October 2019 at the Pedagogical University of Cracow. From that period I was collaborating remotely, due to the COVID pandemic time, with Alexandru Dimca (Professor Emeritus at Nice) and Tim Römer (Professor at Osnabrück). In all cases mentioned above, the collaborations were successful with papers published in recognized mathematical journals, like in *Crelle's journal*, *Journal of Algebraic Combinatorics*, *Manuscripta Mathematica*, or *Results in Mathematics*.

In the meantime, working both in Germany and Poland, I took part in many conferences and I was an invited speaker for several conferences. The most important talks that I gave took place in Oberwolfach, Luminy, and Edinburgh. Moreover, I presented posters at several occasions, probably the most important presentation was during the AMS Summer Institutes - Algebraic Geometry 2015 in Salt Lake City.

## Presentation of teaching and organizational achievements as well as achievements in popularization of science

### *Teaching*

I started my teaching duties in the Winter Semester 2013/2014 when I was conducting my internship at the Warsaw Center for Mathematics and Computer Sciences, and I was running exercises classes in *Linear Algebra I* at the Warsaw University. Then, after my PhD defence, I was running classes for maths students at the Pedagogical University of Cracow, namely *Computer Algebra* and *Introduction to Programming in C++*, and simultaneously I was lecturing *Mathematics for biologists*. During the time in Germany, in Mainz I was teaching *Topologie* (an advanced course, in German) and *Computeralgebra* (both in English and German). After moving to Hannover, I was teaching *Lineare Algebra I* (in German) and *Ebene Kurven - eine konkrete Einführung* (in English and German). In the period of November 2017 - September 2019 I held a position at the Polish Academy of Sciences and I did not conduct teaching duties. After moving to the Pedagogical University of Cracow I resumed my teaching. In this period of the last three years I run classes on every stage of the academic education (both in Polish and in English). I am regularly teaching the whole one year course devoted to *Linear Algebra I & II* (as a tutor running the exercises classes). I am teaching regularly in the Doctoral School at our University (all classes are in English), and in this case I am teaching a general course *Research Grants I & II* and *Workshops in Specialized Mathematics* (12 hours each). Since I have an experience in grant applications (both personal and institutional) my classes are mostly focused on providing the most important features and tricks in applications. In the case of Special Workshops, I am teaching a combination of advanced algebraic geometry, algebraic topology, and complex analysis. I was teaching specialized courses for Bachelor students, for instance *Curves and Surfaces with interesting properties* (two times), and *Vector Bundles*.

Moreover, I supervised 8 Bachelor Theses and 2 Master Theses. Right now I am supervising 4 Master Theses and two Doctoral Students. Among these Bachelor and Master Students, as grades for written theses, 2 of them received the grade A with Distinctions, 6 of them received grade A, and 2 of them received grade B. Moreover, two of these students wrote their first research papers, namely

- Maria Tombarkiewicz (Bachleor Thesis, Grade: **A with Distinctions**): her project was devoted to symbolic powers of ideals associated with point-line configurations in the complex projective plane, based on her thesis, together with another student, she wrote a research paper entitled *On Yoshinaga's arrangement of lines and the containment problem* that has been recently published in **Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie**.
- Aleksandra Gałecka (Master Thesis, Ongoing): her project is devoted to the geometry of nearly-free arrangements of conics and lines in the complex projective plane having only nodes, tacnodes, and ordinary triple points. Her results, providing an almost complete characterization of such arrangements, have been recently published in the paper entitled *On the nearly freeness of conic-line arrangements with nodes, tacnodes, and ordinary triple points* in **Boletín de la Sociedad Matemática Mexicana**.

*Advising*

In 2019-2020 I acted as an Auxiliary Supervisor of mgr Jakub Kabat (Pedagogical University of Cracow), his Main Supervisor was prof. Tomasz Szemberg. Kabat defended his PhD thesis in November 2020, the title of doctoral thesis is *Line arrangements in algebraic terms*, and it is available online on the web-page of the repository of the Pedagogical University of Cracow. His thesis was partially devoted to the classification of free line arrangements with double and triple points, the theory of symbolic powers of homogeneous ideals, and constructing examples of supersolvable line arrangements using Ziegler's trick by extensions.

Moreover, I was asked to take part both as a member of committees and as a referee for two PhD theses, namely

- PhD thesis in Mathematics at the University of Kingston, Chrisa Dionne, thesis *Numerical restrictions on Seshadri curves with applications to* $\mathbb{P}^1 \times \mathbb{P}^1$, defended at 13th October 2021.
- PhD thesis in Mathematics at Leibniz Universität in Hannover, David Geis, thesis *On the combinatorics of Tits arrangements*, defended with distinctions at 31st May 2018.

*Organisation*

In the period 2016-2022 I was co-organizing 6 international conferences, both in Poland and Germany. In 2022, I was co-organizing two satellite conferences to the Virtual ICM, namely MEGA 2022 (Effective Methods in Algebraic Geometry), 20-24 June 2022 in Kraków and Recent Advances in Classical Algebraic Geometry, 28 June - 2 July 2022 in Kraków. I am regularly organizing reading seminar for young researchers at the Pedagogical University of Cracow. As a member of the scientific community at my home university, I am a member of the following bodies:

- The Scientific Council of the Discipline Mathematics,
- The Council of the Institute of Mathematics,
- The Commission for the Scientific Matters at the Pedagogical University of Cracow,
- The Commission for the Realization of the Programme Excellence Initiative - Research University.

Moreover, I also serve as the Deputy Chairman of the Scientific Council of the Discipline Mathematics.

I am also involved in institutional grant applications that are regularly submitted by my home university. For instance, I was in the team which successfully applied for Polish National Agency for Academic Exchange Grant *STER* in 2021 (around 1 400 000 PLN subsidy).

*Popularization of science*

During the time when I was in Poland (2012 - 2015), I was giving regularly lectures at my home secondary school in Mielec for math classes. I am also taking part in students conferences which are organized by the Science Club Mathematics at the Pedagogical University of Cracow. In 2021, I gave a public talk during one such event which was devoted to combinatorial problems of point-line configurations in the real plane. Also, when I was an undergraduate student, I was involved in many activities organized in the framework of *Children's University Foundation*, mostly related with special workshops devoted to mathematical games and problems for children.

## Other informations

*Selected awards and scholarships*

2020 **DAAD Scholarship** Research Stays for University Academics and Scientists, October 2020.

2018 **Distinguished Reviewer of Zentralblatt MATH**, awarded by FIZ Karlsruhe and EMS.

2018 **START 2018 Award** of the Foundation for Polish Science (the most prestigious scholarship for young Polish researchers, awarded yearly to 100 scientists under 30).

2018 **Kazimierz Kuratowski Award** for Mathematicians below 30 years old – the most prestigious prize for young mathematicians in Poland.

2016 **Riemann Scholarship**, Riemann Center for Physics and Geometry Leibniz Universität Hannover, declined.

2016 **Award of the Second Category** from the Polish Ministry of Higher Education and Science for outstanding scientific achievements.

2014 **DAAD Scholarship** for a Short Research Stays in Germany for PhD students, June 2014.

2011–2012 **Scholarship from Małopolska Fundacja Stypendialna "Sapere Auso"**.

2010–2012 **Scholarship of the Polish Minister of Science and Higher Education**.