Professional Accomplishments

1 Name

Justyna Szpond

2 Diplomas, degrees conferred in specific areas of science or arts, including the name of the institution which conferred the degree, year of degree conferment, title of the PhD dissertation

- Doctor of Philosophy in Mathematics,
  Jagiellonian University, Krakow, June 2006,

  Thesis title: *Polynomial ideals in the ring of holomorphic functions*

  Advisor: Tadeusz Winiarski

- Master of Science in Mathematics,
  Pedagogical University of Cracow, June 2002,

  Thesis title: *The Chinese Remainder Theorem*

  Advisor: Tadeusz Winiarski

3 Information on employment in research institutes or faculties/departments or school of arts

- Institute of Mathematics, Polish Academy of Sciences, October 2020 – September 2021
  temporary research position,
- Department of Mathematics, Pedagogical University of Cracow, October 2014 – until now
  associate professor,
- Department of Mathematics, Pedagogical University of Cracow, October 2002 – September 2014
  teaching assistant.

4 Description of the achievements, set out in art. 219 para 1 point 2 of the Act

The boldface number at the end of each item is the current number of citations according to the Web of Science. The number in italic indicates the current number of citations according to Google Scholar. As the articles forming the achievement are quite recent, this seems a relevant information.

Publications building up the achievement

[Hab1] **Szpond, J.**: Unexpected hypersurfaces with multiple fat points, Journal of Symbolic Computation, DOI: 10.1016/j.jsc.2020.07.018, (**WOS: 0**, *GS: 8*)

[Hab2] Bauer, T., Malara, G., Szemberg, T., **Szpond, J.**: Quartic unexpected curves and surfaces, manuscripta math. 161 (2020), 283 – 292 , (**WOS: 6**, *GS: 25*)

[Hab3] **Szpond, J.**: Unexpected curves and Togliatti-type surfaces, Math. Nachr. 293 (2020), 158–168, (**WOS: 2**, *GS: 10*)

[Hab4] **Szpond, J.**: Fermat-type arrangements, In: Stamate D., Szemberg T. (eds) Combinatorial Structures in Algebra and Geometry. NSA 2018. Springer Proceedings in Mathematics & Statistics, vol 331. p. 161 – 182, Springer, Cham, 2020, (**WOS: 0**, *GS: 14*)

[Hab5] Malara, G., **Szpond, J.**: Fermat-type configurations of lines in $\mathbb{P}^3$ and the containment problem, J. Pure Appl. Algebra, **222** (2018), 2323 – 2329, (**WOS: 4**, *GS: 11*)

[Hab6] Malara, G., **Szpond, J.**: On codimension two flats in Fermat-type arrangements, Multigraded algebra and applications, 95 – 109, Springer Proc. Math. Stat., 238, Springer, Cham, 2018, (**WOS: 3**, *GS: 13*)

[Hab7] Lampa-Baczyńska, M., **Szpond, J.**: From Pappus Theorem to parameter spaces of some extremal line point configurations and applications, Geom. Dedicata, 188 (2017), 103 – 121, (**WOS: 5**, *GS: 9*)

4.1 Introduction: Fermat-type arrangements

Arrangements of hyperplanes are classical subject of study in mathematics. They appear in various fields of mathematics: combinatorics (e.g. Peter Orlik and Louis Solomon [36]), commutative algebra (e.g. Isabella Novik, Alexander Postnikov and Bernd Sturmfels [35]), representation theory (e.g. Gustav Lehrer [29]), algebraic geometry (e.g. Xavier Roulleau and Giancarlo Urzúa [38]), analytic spaces (e.g. Pierre Deligne [13]).

Reflection arrangements, because of their rich symmetries, enjoy particularly nice properties. Recall that a linear automorphism $g \in \mathrm{GL}(V)$ of a finite dimensional vector space V is a reflection if $g^n = \mathrm{id}$ for some n and exactly $\dim(V) - 1$ of eigenvalues of g are equal 1. A finite group $G \subset \mathrm{GL}(V)$ is a reflection group if it is generated by reflections. A celebrated result of Shephard and Todd [39] classifies all finite reflection groups. A reflection arrangement is the finite collection of hyperplanes in V determined by all reflections in a reflection group G. Such arrangements have attracted recently considerable attention in some new areas of commutative algebra (the containment problem) and algebraic geometry (the bounded negativity conjecture, special linear systems) as they have been used there to construct instructive examples and counter-examples.

Among reflection groups there is an infinite series of groups denoted by $G(n, n, N + 1)$. The group $G(n, n, N+1) \subseteq \mathrm{GL}(\mathbb{C}^{N+1})$ consists of monomial matrices whose entries are n-th roots of unity subject to the condition that their product is 1. Recall that a matrix is called monomial if in each column and each row there is exactly one non-zero element.

The reflection hyperplanes of $G(n, n, N + 1)$ are given by equations

$$x_i - \varepsilon^\alpha x_j = 0,$$

where ε is a primitive n-th root of unity and $\alpha \in \{0, 1, \ldots, n - 1\}$. Thus they are exactly factors of the Fermat-type polynomial

$$F_{N,n}(x_0 : \ldots : x_N) = \prod_{0 \leqslant i < j \leqslant N} (x_i^n - x_j^n).$$

The numbering of coordinates suggests already that we are interested in viewing the hyperplanes in the projective space $\mathbb{P}^N(V)$. It is convenient to introduce the extended polynomial

$$F_{N,n,N}(x_0 : \ldots : x_N) = x_0 \cdot \ldots \cdot x_N \cdot F_{N,n}(x_0 : \ldots : x_N)$$

as well as intermediate cousins

$$F_{N,n,k}(x_0 : \ldots : x_N) = x_0 \cdot \ldots \cdot x_k \cdot F_{N,n}(x_0 : \ldots : x_N).$$

Definition 1 *The (extended, intermediate resp.) Fermat-type arrangement $\mathcal{F}_N^n$ ($\mathcal{F}_{N,N}^n$, $\mathcal{F}_{N,k}^n$ resp.) is the set of hyperplanes in $\mathbb{P}^N$ given by linear factors of $F_{N,n}$ ($F_{N,n,N}$, $F_{N,n,k}$ resp.).*

Remark 2 *Traditionally the name of the Fermat arrangement is assigned to the arrangement of lines in $\mathbb{P}^2$ defined by*

$$(x^n - y^n)(y^n - z^n)(z^n - x^n) = 0.$$

The name is motivated by the observation that the arrangement is exactly the singular locus of the pencil of curves

$$C_{\lambda:\mu} : \lambda(x^n - y^n) + \mu(y^n - z^n) = 0$$

in which all non-singular members are isomorphic to the Fermat curve

$$x^n + y^n + z^n = 0.$$

For $n = 3$ one gets the famous dual Hesse arrangement determined by lines dual to 3-torsion points on the elliptic curve

$$x^3 + y^3 + z^3 = 0.$$

The way, the Fermat-type arrangements are constructed is a simple illustration of the method which I call conification.

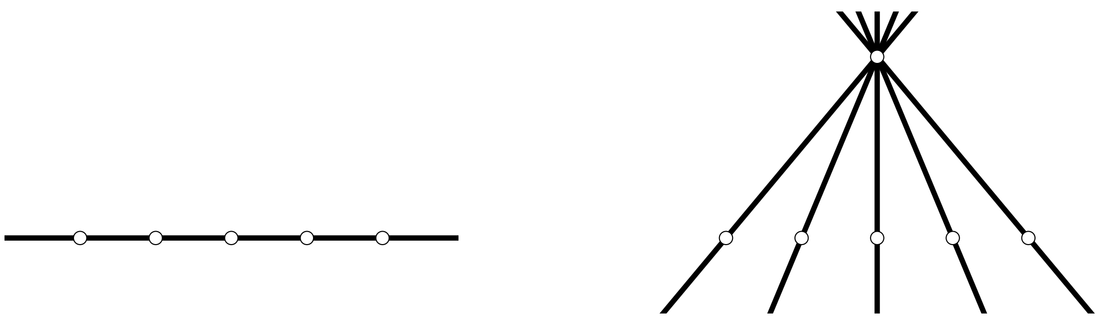
For $N = 1$, the hyperplanes are just points in $\mathbb{P}^1$ given by the equation

$$x_0^n - x_1^n = 0.$$

Hence, they are determined by n-th roots of unity

$$(1:1),\ (1:\varepsilon),\ (1:\varepsilon^2),\ \ldots,\ (1:\varepsilon^{n-1}), \tag{1}$$

where ε is the n-th root of unity. For $n = 5$ these points are indicated in Figure 1a.



(a) Fermat-type arrangement in $\mathbb{P}^1$

(b) The cone in $\mathbb{P}^2$ over $\mathcal{F}_1^5$

Figure 1: Conification of Fermat-type arrangement for $n = 5$

Now, introducing a new coordinate, say x_2, we can embed the points above into $\mathbb{P}^2$ for example as follows

$$(1:1:0),\ (1:\varepsilon:0),\ (1:\varepsilon^2:0),\ \ldots,\ (1:\varepsilon^{n-1}:0).$$

Joining them with the coordinate vertex $(0:0:1)$ we obtain n lines, which are, of course, zero set of the same polynomial

$$x_0^n - x_1^n = 0$$

considered now as a polynomial in variables x_0, x_1 and x_2. For $n = 5$ these lines are indicated in Figure 1b. One can play the same game extending coordinates of points in (1) by the zero at the beginning or in the middle. The union of these cones obtained in this way is the zero locus of the polynomial

$$(x_0^n - x_1^n)(x_1^n - x_2^n)(x_2^n - x_0^n) = 0,$$

which is exactly $F_{2,n}$.

Passing from $\mathbb{P}^2$ to $\mathbb{P}^3$, we consider again four cones with vertices at the coordinate points in $\mathbb{P}^3$ over Fermat arrangements of lines in the complimentary plane. Every cone consists of n planes. Each of these planes is generated by the vertex of the cone and a line in the complementary plane. And so on.

It is worth to mention that in the special case $n = 1$ the polynomial $F_{N,1}$ is the Vandermonde determinant

$$F_{N,1}(x_0 : \ldots : x_N) = \prod_{0 \leqslant i < j \leqslant N} (x_i - x_j) = \det \begin{pmatrix} 1 & x_0 & \ldots & x_0^N \\ 1 & x_1 & \ldots & x_1^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \ldots & x_N^N \end{pmatrix}.$$

The corresponding (very simple) arrangement is known as the braid arrangement. It plays an important role in algebraic combinatorics.

Passing to the achievement, my research in recent years was motivated by two main subjects which gained considerable attention in algebraic geometry and commutative algebra:

- the containment problem and
- the existence of unexpected hypersurfaces.

In both these areas of study arrangements of hyperplanes in projective spaces play a prominent role. My key contribution to the subject and the concept unifying both research directions is the idea of systematic study of Fermat arrangements of hypersurfaces and derived objects they define. By these I mean the configurations of flats cut out by Fermat arrangement hypersurfaces, their ideals, some alterations of these ideals and configurations of points dual to the hyperplanes. These objects are present in one or another guise in all articles collected in the presented achievement and the idea of studying them has led to substantial progress in both research areas.

I give an introduction to both circles of ideas below and put my results in the perspective of their development. However, before turning to details, let me highlight briefly main results of articles forming the achievement.

Articles [Hab5] and [Hab6] use flats cut out by hyperplanes in Fermat-type arrangements in $\mathbb{P}^3$, respectively in $\mathbb{P}^N$ for $N \geqslant 3$ in order to illustrate that the containment $I^{(3)} \subseteq I^2$ may fail for ideals supported on flats in codimension 2 in projective spaces. This generalizes earlier results of this type for points in $\mathbb{P}^2$.

Article [Hab7] revolves also around the containment problem. Whereas most Fermat-type arrangements cannot be realized over the real number, we show that certain line arrangements introduced by Böröczky admit 1-dimensional space of deformations. We were very lucky to study the Böröczky arrangement of 12 lines since it seems the only case which admits deformations defined over $\mathbb{Q}$. Thus [Hab7] provides the first non-containment result for $I^{(3)}$ and I^2 for an ideal of points with rational coordinates.

Paper [Hab4] studies Fermat-type arrangements from the point of view of the containment problem as well as from the point of view of unexpected hypersurfaces. In particular it contains the first example of an infinite series of unexpected curves where the multiplicity of the general point is fixed at 4, whereas the degree of unexpected curves grows to infinity. This example is strongly related to Fermat arrangements of lines. [Hab4] systemizes also and completes some classification results concerning Fermat-type arrangements scattered in the literature.

In [Hab2] we provide the first example of an unexpected surface in $\mathbb{P}^3$, before the problem was studied only for curves in $\mathbb{P}^2$. We provide also explicit, surprisingly simple equations of some unexpected curves and surfaces and initiate the study of what in now called the BMSS duality.

This idea is considerably extended in [Hab3], which establishes direct links between the theory of unexpected hypersurfaces, Lefschetz-type properties and higher osculating spaces of embedded varieties. The article provides detailed study of all these phenomena for the arrangement of lines determined by the $B3$ root system, which is a special case of an extended Fermat arrangement.

Finally, [Hab1] combines Fermat-type arrangements and the conification method in order to provide the first (an apparently up to the date of this writing the only) example of unexpected hypersurfaces with multiple fat points. More precisely, the existence of such examples is established theoretically in $\mathbb{P}^5$, by computer calculations in $\mathbb{P}^7$ and $\mathbb{P}^9$ and it contains a conjectural results on unexpected hypersurfaces with k general fat points in $\mathbb{P}^{2k+1}$ for $k \geqslant 5$.

4.2 The containment problem

In this part I will outline rudiments of the theory. My join work with Szemberg [42] provides a much more detailed introduction.

In the setting relevant here it suffices to restrict to the ring of polynomials. Let $\mathbb{K}$ be a field and let I be a homogeneous ideal in the polynomial ring $R = \mathbb{K}[x_0, \ldots, x_N]$. There are two families of ideals naturally associated to I. The first is given by ordinary (algebraic) powers of I

$$R \supseteq I \supset I^2 \supset I^3 \supset I^4 \supset \ldots . \tag{2}$$

The other is provided by symbolic powers of I

$$R \supseteq I \supset I^{(2)} \supset I^{(3)} \supset I^{(4)} \supset \ldots . \tag{3}$$

Recall that for a positive integer m, the m-th symbolic powers $I^{(m)}$ of I is defined as

$$I^{(m)} = \bigcap_{P \in \mathrm{Ass}(I)} (R \cap (I^m)_P),$$

where the intersection takes place in the field of fractions of R and $\mathrm{Ass}(I)$ denotes the set of associated primes of I. The leading question here is if there is any regularity in the containments between members of each family not depending on the particular choice of I.

While it is easy to see that the containment $I^r \subseteq I^{(m)}$ holds if and only if $r \geqslant m$, the reverse containment is much more intriguing.

Problem 3 (Containment Problem) *Fixing R, determine for which m and r there is the containment*

$$I^{(m)} \subseteq I^r$$

for all proper ideals $I \subset R$.

The possibility that a regularity exists also in this direction was put forwards by Swanson in [41]. At the beginning of this Millenium two independent works appeared to the effect that the following result holds.

Theorem 4 (Ein-Lazarsfeld-Smith and Hochster-Huneke) *Let $I \subseteq \mathbb{K}[x_0, \ldots, x_N]$ be an arbitrary homogeneous ideal. Then*

$$I^{(m)} \subseteq I^r \tag{4}$$

holds for all $m \geqslant N \cdot r$.

Ein, Lazarsfeld and Smith in [19] proved Theorem 4 in characteristic 0 using multiplier ideals, whereas Hochster and Huneke in [27] worked in positive characteristic using tight closures. Recently Theorem 4 has been generalized to mixed characteristic by Ma and Schwede in [30]. Interestingly, not a single example showing that the bound in (4) is sharp is known. Usually the containment holds for smaller values of m. The so called star configurations, see [22], provide a series of examples where $m \geqslant Nr - (N-1)$ is necessary. This observation prompted Huneke to ask if it is true for radical ideals I of points in $\mathbb{P}^2$ that the containment $I^{(3)} \subseteq I^2$ holds. Parallel to that Ein asked if there is a simple (not appealing to multiplier ideals and Skoda-type theorems) proof for the $I^{(4)} \subseteq I^2$ containment. The question of Huneke has been taken on by Harbourne. His works on the subject, notably with Bocci [9], [10], culminated in the joint work with Huneke where they pose the following problem.

Problem 5 (see [24, Conjecture 4.1.1]) *Let I be a homogeneous ideal in the ring of polynomials. Assume that the set of zeroes of I consists of finitely many points in the projective space $\mathbb{P}^N$. Is then*

$$I^{(m)} \subseteq I^r$$

for all $m \geqslant Nr - (N-1)$?

The first non-trivial case is that of $N = r = 2$, which is exactly the $I^{(3)} \subseteq I^2$ question of Huneke.

In 2013 Dumnicki, Szemberg and Tutaj-Gasińska announced in [18] the first non-containment example. They showed that for the ideal I of 12 intersection points of the dual Hesse arrangement there is

$$I^{(3)} \not\subseteq I^2.$$

This ideal is an almost complete intersection and in convenient coordinates it is generated by the following polynomials

$$I = \left\langle x(y^3 - z^3), y(z^3 - x^3), z(x^3 - y^3) \right\rangle.$$

The points in I are exactly the intersection points of the arrangement of lines defined by linear factors of the polynomial

$$F_{2,3}(x, y, z) = (x^3 - y^3)(y^3 - z^3)(z^3 - x^3). \tag{5}$$

This is exactly where arrangements come into the picture. The non-containment example prompted a number of new problems going in various directions.

Firstly, since the arrangement formed by factors of (5) is not defined over the real numbers, it was asked if there are such examples over the reals or even over the rationals. Harbourne and Seceleanu provided a long list of non-containment results in finite characteristic in [26].

First examples of containment failure over the reals were found by combined efforts of scientists from the Jagiellonian and the Pedagogical Universities in Cracow [12]. The first non-containment examples over rationals have been extensively studied in my joint paper with Lampa-Baczyńska [Hab7]. All non-containment examples mentioned so far are provided by ideals of points which are intersection points of some symmetric arrangement of lines.

The arrangement determined by (5) is called the dual Hesse arrangement. There are 9 lines in the arrangement, they intersect in triples in altogether 12 points and there are no other intersection points between the lines. By the well-known Sylvester-Gallai Theorem such an arrangement cannot be realized over the real numbers. It is also well-known that all arrangements of 9 complex lines with 12 triple intersection points are projectively equivalent to the arrangement defined by (5).

The first example of the non-containment $I^{(3)} \not\subseteq I^2$ over the real numbers is provided by Böröczky arrangement of 12 lines, which is visualized in Figure 2. The role of the circle is explained in the description of construction below.

This arrangement is not defined over the rational numbers. More precisely, Böröczky arrangements in the real affine plane are constructed as follows.

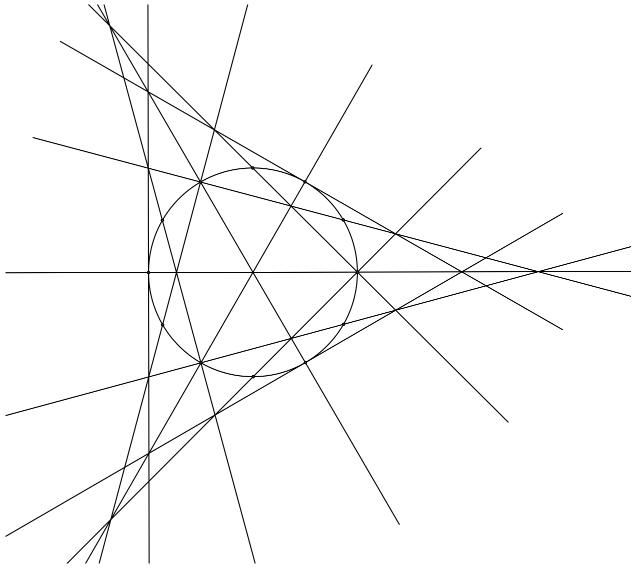


Figure 2: Böröczky arrangement of 12 lines

Let $\mathcal{C}$ be the unit circle centered in the origin, i.e., $\mathcal{C}$ is the zero set of the polynomial $\mathcal{C}(x, y) = x^2+y^2-1$. Let $P(\alpha) = (\cos\alpha, \sin\alpha)$ denote a point on $\mathcal{C}$ and let $L(\alpha)$ be the line joining points $P(\alpha)$ and $P(\pi - 2\alpha)$. If these points coincide, we take as $L(\alpha)$ the tangent line to $\mathcal{C}$ at $P(\alpha)$.

Definition 6 *Let $n \geqslant 3$ be an integer. The Böröczky arrangement B_n consists of lines*

$$B_n = \left\{ L(0), L\left(\frac{1}{n}2\pi\right), \ldots, L\left(\frac{n-1}{n}2\pi\right) \right\}.$$

Böröczky examples were introduced in connection with the Dirac-Motzkin Conjecture which stipulates that if $\mathcal{L} = \{L_1, \ldots, L_s\}$ is an arrangement of s lines in the real affine (or projective) plane, not forming a pencil, then there are at least $\frac{s}{2}$ points where exactly 2 arrangements lines intersect (double points). Böröczky examples have the number of double points close to this lower bound. Notably the Conjecture has been solved for large s only recently by Green and Tao [23].

4.2.1 New construction of deformations of the B_{12} arrangement

In our joint work with Lampa-Baczyńska [Hab7] we realized that Böröczky arrangements are not rigid, they admit certain deformations which are parametrized by an algebraic curve. The construction of the parameter space for B_{12} arrangements presented below is a modification of that presented in [Hab7] and therefore carried out here in some detail. On the other hand, for convenience of the reader, I decided to keep the notation from [Hab7]. This is why points appear not in the alphabetical order. Figure 3 is taken from [Hab7]. Constructions of parameter spaces for Böröczky arrangements with higher number of lines are similar and I do not repeat them here, passing rather straight to Proposition 9.

Since any two lines in projective plane are projectively equivalent we may assume, see Figure 3, that points F, D, L and E are as follows

$$F = (1:0:0),\ D = (0:1:0),\ L = (0:0:1),\ E = (1:1:1).$$

Then we have the following equations of lines

$$EF: y - z = 0,\ DE: x - z = 0,\ EL: x - y = 0.$$

These 4 points and 3 lines are the initial data of our construction. The resulting arrangement is not completely determined by this data. It is necessary to choose an additional point

$$A = (a : a : b)$$

on the line EL different from points E and L, i.e. $a \neq b$ and $a \neq 0$. The choice of $(a : b)$ determines the rest of construction. The key point here is that the construction gives an arrangement which inherits all of the combinatorics of B_{12} for all values $(a : b) \in \mathbb{P}^1 \setminus \{(0 : 1), (1 : 1), (1 : 0)\}$. Taking lines through available points and intersecting resulting lines to determine new points step by step the whole arrangement is constructed.

For example, the first two steps of this procedure are the lines

$$\begin{aligned} AF &: by - az = 0, \\ AD &: bx - az = 0. \end{aligned}$$

Knowing them, we can create new intersection points

$$\begin{aligned} J = AF \cap ED &= (b : a : b), \\ K = AD \cap EF &= (a : b : b), \end{aligned}$$

which, in particular imposes the condition $b \neq 0$.

Continuing, we construct lines JL, KL, points H, I, P, N, lines FI, DH, points C, G, B, O, M, lines GN, CP, and points Q, R, S. In this way we obtain all points in the configuration B_{12}. Adding the line MO, we have also all lines. The construction is complete but it requires a proof that indeed all required incidences are fulfilled. More precisely, since points are created as intersection points of two lines, it is necessary to check that they lie on an additional line. By construction, it is required to verify the following incidences:

$$B \in EL,\ S \in EL,\ Q \in MO,\ R \in MO.$$

In the paper [Hab7] we show, using the Pappus Theorem, that these incidences hold without any additional restrictions on the values of a and b.

We complete these considerations by listing the coordinates of all relevant points:

$$\begin{array}{rclrcl} H &=& (b : a : a), & I &=& (a : b : a), \\ P &=& (a^2 : ab : b^2), & N &=& (ab : a^2 : b^2), \\ C &=& (a^2 : b^2 : ab), & G &=& (b^2 : a^2 : ab), \\ B &=& (b : b : a), & O &=& (b^2 : ab : a^2), \\ M &=& (ab : b^2 : a^2), & Q &=& (b^2 + ab - a^2 : a^2 : a^2), \\ R &=& (a^2 : b^2 + ab - a^2 : a^2), & S &=& (a^2 : a^2 : b^2 + ab - a^2). \end{array}$$

Corollary 7 (compare [Hab7, Theorem A]) *Deformations of the B_{12} arrangement are parametrized by the projective line $\mathbb{P}^1$. Points $(1 : 1)$, $(0 : 1)$ and $(1 : 0)$ correspond to degenerate arrangements, where lines or points fall together.*

As a consequence we conclude the following Theorem.

Theorem 8 (see [Hab7, Theorem A]) *There exists an arrangement of 12 lines defined over $\mathbb{Q}$ with 19 triple and 9 double points (as in Böröczky arrangement).*

Indeed, it suffices to take rational values of the parameters $(a : b)$ different from values corresponding to the degenerated cases listed in Corollary 7. The ideal I of the 19 triple points has the property that

$$I^{(3)} \not\subseteq I^2.$$

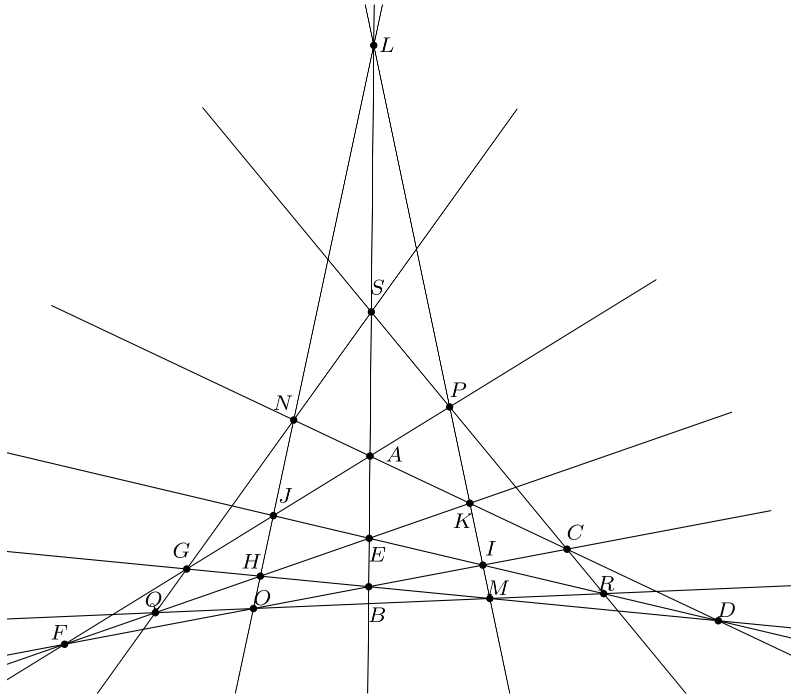


Figure 3: A deformation of the B_{12} arrangement.

As usual, the element contained in $I^{(3)}$ but not contained in I^2 is the product of equations of all lines in the arrangement.

Our discovery in [Hab7] that Böröczky arrangements vary in positive dimensional families led to more research on these families, see [20, 21]. Explicit equations of parameter spaces have been given for Böröczky arrangements of 13, 14, 15, 16, 18 and 24 lines.

Proposition 9 *Let C_n denote the parameter space of the B_n arrangements. Then we have $C_{12} \cong \mathbb{P}^1$. For $n \in \{13, 14, 15, 16, 18, 24\}$ parameter spaces are curves in $\mathbb{P}^1(a:b) \times \mathbb{P}^1(c:d)$ given by the following equations*

n	equation	genus
13	$a^4d^2 - a^3bcd + a^2b^2cd - a^2b^2d^2 + b^4c^2 - 2ab^3cd + 2ab^3d^2 - b^4cd = 0$	2
14	$2ab^2d^2 - 3a^2bd^2 + a^3d^2 + ab^3cd - b^3c^2 = 0$	1
15	$a^4cd - a^2b^2c^2 - a^3bd^2 + a^2b^2cd - ab^3c^2 + b^4c^2 = 0$	1
16	$a^4c^2d - 2a^3bcd^2 - 2a^2b^2c^2d + a^2b^2cd^2 + 2ab^3c^2d - b^4c^3 + a^2b^2d^3 + 2b^4c^2d - 2b^4cd^3 = 0$	2
18	$a^3b^3c^5 - a^5bc^2d^3 + a^4b^2c^3d^2 - 6a^3b^3c^4d + a^2b^4c^5 + a^6d^5 + a^4b^2c^2d^3 + 12a^3b^3c^3d^2 - 4a^2b^4c^4d - 5a^5bd^5 + 7a^4b^2cd^4 - 22a^3b^3c^2d^3 + 11a^2b^4c^3d^2 - ab^5c^4d + 6a^4b^2d^5 - a^3b^3cd^4 + 3a^2b^4c^2d^3 - 4ab^5c^3d^2 + b^6c^4d - 4a^3b^3d^5 + 4a^2b^4cd^4 - ab^5c^2d^3 = 0$	2
24	$a^8c^3d + a^7bc^3d + a^6b^2c^4 - 6a^7bc^2d^2 + 3a^6b^2c^3d - 6a^6b^2c^2d^2 - 2a^5b^3c^3d + 10a^6b^2cd^3 - 6a^5b^3c^2d^2 - 2a^4b^4c^3d - a^6b^2d^4 + 12a^5b^3cd^3 + 3a^4b^4c^2d^2 - 2a^3b^5c^3d - 6a^5b^3d^4 + 3a^4b^4cd^3 + 6a^3b^5c^2d^2 - a^2b^6c^3d - 3a^4b^4d^4 - 13a^3b^5cd^3 + 9a^2b^6c^2d^2 - ab^7c^3d + 4a^3b^5d^4 - 12a^2b^6cd^3 + 6ab^7c^2d^2 - b^8c^3d + 5a^2b^6d^4 - 3ab^7cd^3 + 2ab^7d^4 - b^8cd^3 = 0$	5

Table 1: Equations of parameter spaces for B_n arrangements

The genus in Table 1 refers to the geometric genus of the normalization.

All curves presented in Proposition 9 have a finite number of rational points, which correspond to degenerate cases, where lines or points fall together. A surprising conclusion is that B_{12} seems to be the only Böröczky arrangement which can be realized over the rational numbers.

Jakub Kabat in his PhD thesis, defended in 2020, verified that no Böröczky arrangement with less than 12 lines provides a non-containment example.

It is worth to mention that the first rational simplicial arrangement of 25 lines leading to the non-containment example was described by Malara and myself in [31]. It is visualized in Figure 4. Another example of this kind was presented by Janasz, Lampa-Baczyńska and Malara in [28].

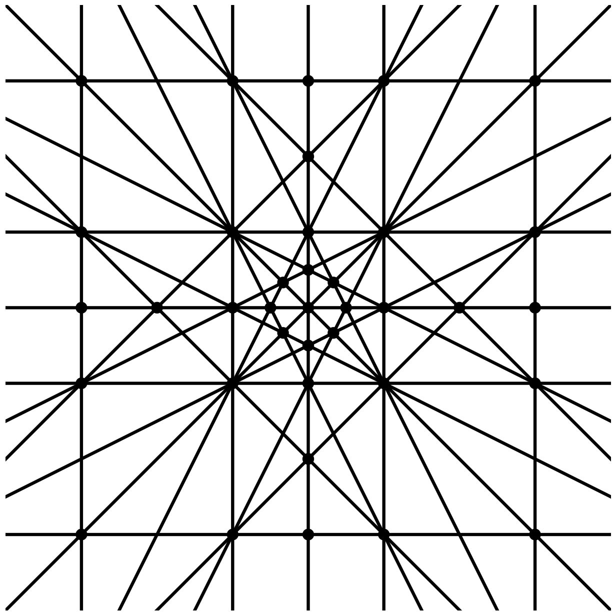


Figure 4: The rational simplicial arrangement of 25 lines, one line is the line at the infinity

4.2.2 Containment problem for linear flats

Theorem 4 can be stated more precisely in the following way. Recall that the big height $\text{bight}(I)$ of an ideal I is the maximal height of its associated primes, i.e., geometrically it is the maximal codimension of an embedded component of I.

Theorem 10 *Let $I \subseteq \mathbb{K}[x_0, \dots, x_N]$ be an arbitrary homogeneous ideal. Then*

$$I^{(m)} \subseteq I^r$$

holds for all $m \geqslant h \cdot r$, where $h = \text{bight}(I)$.

Thus for example for ideals of curves in $\mathbb{P}^3$ the numerology is the same as for ideals of points in $\mathbb{P}^2$. In [Hab5] with Malara we constructed an arrangement of lines in $\mathbb{P}^3$, such that for their defining ideal I the containment $I^{(3)} \subseteq I^2$ fails. Our construction was motivated by the Fermat arrangements of lines in $\mathbb{P}^2$.

Definition 11 (The Fermat arrangement of lines) *The Fermat arrangement of lines $\mathcal{F}_2^n$ in $\mathbb{P}^2$ is defined by linear factors of the following polynomial:*

$$F_{2,n}(x, y, z) = (x^n - y^n)(x^n - z^n)(y^n - z^n).$$

Note that $\mathcal{F}_2^n$ is denoted by $\mathcal{A}_3^0(n)$ by Orlik-Terao [37, Example 6.29 and page 247].

Fermat arrangements of lines are relevant in the containment problem. Note that for $n = 3$, we obtain exactly the arrangement (5) studied by Dumnicki, Szemberg and Tutaj-Gasińska. It was proved in [26, Proposition 2.1] by Harbourne and Seceleanu that for all $n \geqslant 3$ for the ideal I of all intersection points of lines in the Fermat arrangement $\mathcal{F}_2^n$ the containment $I^{(3)} \subseteq I^2$ fails.

During the workshop on Ordinary and Symbolic Powers of Ideals held in the International Research Station in Oaxaca in 2017, I suggested to work with the following generalization: consider arrangement of planes in $\mathbb{P}^3$ defined by vanishing of the polynomial

$$F_{3,n}(x,y,z,w) = (x^n - y^n)(x^n - z^n)(x^n - w^n)(y^n - z^n)(y^n - w^n)(z^n - w^n).$$

There are $6n$ planes in the arrangement. They intersect in pairs along $3n^2$ lines, in triples along $4n^2$ lines and there are 6 additional lines (the edges of the coordinate tetrahedron) of multiplicity n. Since we are interested in the third symbolic power of an ideal, we restrict our attention to those lines which have multiplicity at least 3.

Definition 12 (The restricted Fermat configuration of lines, [Hab5, Definition 3.1]) *The restricted Fermat configuration $\mathcal{RF}_3^n(1)$ of lines in $\mathbb{P}^3$ is the union of all lines, determined by Fermat configuration of planes, with multiplicity at least* 3.

This generalization provided the first (and apparently the only) series of non-containment results for curves in $\mathbb{P}^3$. More precisely our main result from [Hab5] is

Theorem 13 (Lines in $\mathbb{P}^3$, [Hab5, Theorem 4.3]) *For an arbitrary integer $n \geqslant 3$ and $I = I(\mathcal{RF}_3^n(1))$ the ideal of lines in the restricted Fermat arrangement $\mathcal{RF}_3^n(1)$ of lines in $\mathbb{P}^3$ the containment*

$$I^{(3)} \subseteq I^2$$

fails.

The idea of the proof is the following. First, the polynomial $F_{3,n}(x,y,z,w)$ clearly lies in $I^{(3)}$ as it vanishes to order $\geqslant 3$ along all the arrangement lines. The issue is to show that it does not belong to the second ordinary power of I. This is achieved as the high symmetry of the arrangement is mirrored by the symmetry of generators of I, which can be taken as follows

$$\begin{aligned} g_1 &= (x^n - y^n)(z^n - w^n)xy, \quad & g_2 &= (x^n - y^n)(z^n - w^n)zw, \\ g_3 &= (x^n - z^n)(y^n - w^n)xz, \quad & g_4 &= (x^n - z^n)(y^n - w^n)yw, \\ g_5 &= (x^n - w^n)(y^n - z^n)xw, \quad & g_6 &= (x^n - w^n)(y^n - z^n)yz. \end{aligned}$$

Then I^2 is generated by degree 2 monomials in g_i's. Assume to the contrary that

$$F_{3,n} = \sum_{1 \leqslant i \leqslant j \leqslant 6} h_{i,j} g_i g_j. \tag{6}$$

Taking the identity (6) modulo (x) we obtain

$$\begin{aligned} -y^n z^n w^n (y^n - z^n)(y^n - w^n)(z^n - w^n) = {} & y^{2n} z^2 w^2 (z^n - w^n)^2 \widetilde{h}_{2,2} \\ & + z^{2n} y^2 w^2 (y^n - w^n)^2 \widetilde{h}_{4,4} \\ & + w^{2n} y^2 z^2 (y^n - z^n)^2 \widetilde{h}_{6,6} \\ & + y^{n+1} z^{n+1} w^2 (y^n - w^n)(z^n - w^n) \widetilde{h}_{2,4} \\ & + y^{n+1} w^{n+1} z^2 (y^n - z^n)(z^n - w^n) \widetilde{h}_{2,6} \\ & + z^{n+1} w^{n+1} y^2 (y^n - z^n)(y^n - w^n) \widetilde{h}_{4,6}. \end{aligned} \tag{7}$$

We write here $\widetilde{f}$ to indicate the residue class of a polynomial $f \in \mathbb{K}[x,y,z,w]$ modulo (x).

Analysing the coefficient at the monomial $y^{3n}z^{2n}w^n$ on both sides of (7) we conclude that its coefficient in the polynomial $h_{4,4}$ equals -1.

Taking in turn the identity (6) modulo (z) we obtain

$$\begin{aligned} -x^n y^n w^n (x^n - y^n)(x^n - w^n)(y^n - w^n) = {} & w^{2n}x^2y^2(x^n - y^n)^2\widehat{h}_{1,1} \\ & + x^{2n}y^2w^2(y^n - w^n)^2\widehat{h}_{4,4} \\ & + y^{2n}x^2w^2(x^n - w^n)^2\widehat{h}_{5,5} \\ & - x^{n+1}w^{n+1}y^2(x^n - y^n)(y^n - w^n)\widehat{h}_{1,4} \\ & - y^{n+1}w^{n+1}x^2(x^n - y^n)(x^n - w^n)\widehat{h}_{1,5} \\ & + x^{n+1}y^{n+1}w^2(x^n - w^n)(y^n - w^n)\widehat{h}_{4,5}. \end{aligned} \tag{8}$$

We write here $\widehat{f}$ to indicate the residue class of a polynomial $f \in \mathbb{K}[x,y,z,w]$ modulo (z).

Comparing the coefficients in (8) at the same monomial as above, we arrive to the conclusion that in the polynomial $h_{4,4}$ it is equal 1, which gives the desired contradiction.

It is then natural to study the same problem in arbitrary dimension N. Let

$$F_{N,n}(x_0,\ldots,x_N) = \prod_{0 \leqslant i < j \leqslant N} (x_i^n - x_j^n)$$

and let $\mathcal{F}_N^n$ be the arrangement of hyperplanes in $\mathbb{P}^N$ determined by linear factors of $F_{N,n}$. We are interested in the codimension 2 skeleton $\mathcal{F}_N^n(N-2)$ of the arrangement, i.e., the union of codimension 2 linear subspaces of $\mathbb{P}^N$ cut out by hyperplanes forming $\mathcal{F}_N^n$. In fact, we are interested in those codimension 2 linear subspaces where at least 3 arrangement hyperplanes intersect. As in Definition 12 we denote this union by $\mathcal{RF}_N^n(N-2)$, the restricted arrangement of codimension 2 linear subspaces (see [Hab6], section 3). The following result proved with Malara generalizes Harbourne-Seceleanu result and our Theorem 13.

Theorem 14 (Codimension 2 flats, [Hab6, Theorem 1]) *Let I be the ideal of $\mathcal{RF}_N^n(N-2)$. Then*

$$I^{(3)} \not\subseteq I^2.$$

Again, the element which is contained in $I^{(3)}$ but is not in I^2 is the polynomial $F_{N,n}$. The argument is considerably more involved than in the case $N = 3$. The key geometrical observation here is that there is a partition of configuration $\mathcal{RF}_N^n(N-2)$ in $(N+1)$ cones, whose vertices are the coordinate points in $\mathbb{P}^N$. This perspective allows to represent the ideal I as an intersection of $(N+1)$ ideals whose structure is considerably simpler. On the algebraic side our considerations are alleviated by introducing a specific bracket algebra, which satisfies a number of useful properties:

$$[x_{i_1}\ldots x_{i_k}] := \prod_{p<q}(x_{i_p}^n - x_{i_q}^n).$$

Thus, in particular,

$$F_{N,n} = [x_0 \ldots x_N].$$

It is convenient to write $[x_{i_0}\ldots\widehat{x_{i_j}}\ldots x_{i_k}]$ for $[x_{i_0}\ldots x_{i_{j-1}}x_{i_{j+1}}\ldots x_{i_k}]$. The following Lemma collects a number of facts proved in [Hab6].

Lemma 15 ([Hab6, Lemma 1, Lemma 2, Lemma 3, Lemma 4]) *Let $2 \leqslant k \leqslant N$ be an integer, then*

- $[x_{i_0}\ldots x_{i_{k-1}}x_{i_k}] = [x_{i_0}\ldots x_{i_{k-1}}]\prod_{j=0}^{k-1}[x_{i_j}x_{i_k}]$,

- $[x_{i_0} \dots x_{i_k}] = \sum_{j=0}^{k} (-1)^{j+k} x_{i_0}^n \dots \widehat{x_{i_j}^n} \dots x_{i_k}^n [x_{i_0} \dots \widehat{x_{i_j}} \dots x_{i_k}]$,
- *for any* $u \in \{0, \dots, N\}$ *there is*

$$[x_{i_0} \dots x_{i_k}] = \sum_{j=0}^{k} [x_{i_0} \dots x_{i_{j-1}} x_u x_{i_{j+1}} \dots x_{i_k}],$$

- *for auxiliary variables* $y_1, \dots, y_k$ *we have*

$$[x_0 \dots x_k] = \sum_{j=0}^{k} (-1)^j [x_0 \dots \widehat{x_j} \dots x_k][x_j y_1] \dots [x_j y_k].$$

Lemma 15 makes explicit calculations involving generators of I feasible.

Proposition 16 ([Hab6, Proposition 1]) *Keeping the notation from Theorem 14 for integers* $N \geqslant 2$ *and* $n \geqslant 3$.
a) If $N = 2M$ *is an even number, let* $A = \{i_1, \dots, i_M\}$ *be a subset of* M *elements in the set* $\{0, 1, \dots, N\}$ *and let* $B = \{j_0, \dots, j_M\}$ *be the complimentary set. The ideal* I *is generated by all polynomials of the form*

$$g_A = x_{i_1} \dots x_{i_M} [x_{i_1} \dots x_{i_M}][x_{j_0} \dots x_{j_M}].$$

b) If $N = 2M + 1$ *is an odd number, let* $A = \{i_0, \dots, i_M\}$ *be a subset of* $M + 1$ *elements in the set* $\{0, 1, \dots, N\}$ *and let* $B = \{j_0, \dots, j_M\}$ *be the complimentary set. The ideal* I *is generated by all polynomials of the form*

$$g_A = x_{i_0} \dots x_{i_M} [x_{i_0} \dots x_{i_M}][x_{j_0} \dots x_{j_M}].$$

This Proposition accompanied by tedious calculations using properties listed in Lemma 15 leads to the proof of Theorem 14. The details are presented in [Hab6].

4.3 The existence of unexpected hypersurfaces

A fundamental objects of study in algebraic geometry are linear systems of divisors. A basic question concerning them is the determination of their dimension. Here I will focus on linear systems of hypersurfaces in projective spaces with finitely many assigned base points of given multiplicities. I will be also mainly interested in general points, even if the considered linear systems are determined in first instance by some symmetric configurations of points. The central question here can be stated as follows.

Problem 17 *Given a linear system* W *of hypersurfaces in projective space and finitely many general points with assigned multiplicities at each point, what is the dimension of the subsystem of* W *formed by all elements of* W *having at the given points at least the assigned multiplicities?*

This elementary stated problem has been considered, in one form or another, since beginnings of this area of mathematics. It can be tracked back to the 18th century to works of Étienne Bézout. In the 19th century it was presented in works of Julius Plücker, Luigi Cremona, Max Noether, Eugenio Bertini, Corrado Segre and others. Important contributions in 20th century go back to Guido Castelnuovo, Federigo Enriques, Francesco Severi, Alessandro Terracini, Benjamin Segre, Andre Hirschowitz, Brian Harbourne, Robert Lazarsfeld, Ciro Ciliberto and many others. Motivations to its study come from different fields of mathematics, see e.g. [34], and answers to the problem find applications in other areas of mathematics, see e.g. [32].

The problem is unsolved in its generality. In fact there is a number of more specific questions and conjectures which remain unsolved for many years. Let me mention here just one of these conjecture due to Nagata [34].

Conjecture 18 (Nagata, 1959) *Let $P_1, \dots, P_s$ be $s \geqslant 10$ general points in $\mathbb{P}^2$. Let C be a curve of degree d passing through all these points with multiplicities at least m. Then*

$$d > m\sqrt{s}.$$

In particular, Nagata's Conjecture asserts that the linear system of plane curves of degree d vanishing to order at least m at s general points in $\mathbb{P}^2$ is empty for $d \leqslant m\sqrt{s}$. Even this is not known for s not a perfect square.

If the assigned multiplicities of points are small, then we can say something more. For example, let $Z = \{P_1, \dots, P_s\}$ be a set of general points in $\mathbb{P}^N$. Then

$$h^0\left(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z)\right) = \max\left\{0, \binom{d+N}{N} - s\right\}.$$

In other words, general points with multiplicities 1 always impose independent conditions on hypersurfaces of arbitrary degree d in the projective space of arbitrary dimension N.

A similar result for points of multiplicities 2 is due to Alexander and Hirschowitz and it is a culmination of two decades of their work.

Theorem 19 (Alexander, Hirschowitz) *Let Z be a set of s points in $\mathbb{P}^N$. Then*

$$h^0\left(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z)^{(2)}\right) = \max\left\{0, \binom{N+d}{N} - s(N+1)\right\},$$

except in the following cases

- $d = 2, 2 \leqslant s \leqslant N$;
- $N = 2, d = 4, s = 5$;
- $N = 3, d = 4, s = 9$;
- $N = 4, d = 4, s = 14$;
- $N = 4, d = 3, s = 7$.

There is not even a conjectural picture for points of multiplicities 3 or more. Going in a somewhat different direction, it is easy to see that a single point imposes independent conditions, regardless of the assigned multiplicity:

$$h^0\left(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(P)^{(m)}\right) = \max\left\{0, \binom{N+d}{N} - \binom{N+m-1}{N}\right\}.$$

Around 2013 Di Gennaro, Ilardi and Vallès [14] discovered a very surprising fact that the above statement may fail for *incomplete* linear series. Let $W \subseteq |\mathcal{O}_{\mathbb{P}^N}(d)|$ be a linear subsystem of hypersurfaces of degree d. Then it might happen that

$$\dim |W \otimes I(P)^m| > \max\left\{\dim W - \binom{N+m-1}{N}, -1\right\}. \tag{9}$$

The first example of this phenomenon is constructed as follows. Let Z be the set of points:

$$\begin{array}{lll} P_1 = (1:0:0), & P_2 = (0:1:0), & P_3 = (0:0:1), \\ P_4 = (1:1:0), & P_5 = (1:-1:0), & P_6 = (1:0:1), \\ P_7 = (1:0:-1), & P_8 = (0:1:1), & P_9 = (0:1:-1), \end{array} \tag{10}$$

in $\mathbb{P}^2$ and let $W = |H^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4) \otimes I(Z)\right)|$. It is elementary to check that $\dim W = 5$. Thus with $N = 2$ and $m = 3$ the right hand side of (9) is -1 and we do not expect a quartic curve vanishing at Z and vanishing to order 3 at $P = (a : b : c)$ general. However such a quartic exists! The existence was proved in [14]. In [Hab2] we provided its explicit equation

$$\begin{aligned} f_P(x : y : z) = 3a(b^2 - c^2) \cdot x^2yz + 3b(c^2 - a^2) \cdot xy^2z + 3c(a^2 - b^2) \cdot xyz^2 \\ + a^3 \cdot y^3z - a^3 \cdot yz^3 + b^3 \cdot xz^3 - b^3 \cdot x^3z + c^3 \cdot x^3y - c^3 \cdot xy^3. \end{aligned} \tag{11}$$

It is visualized in Figure 5. The doted points are the points in (10).

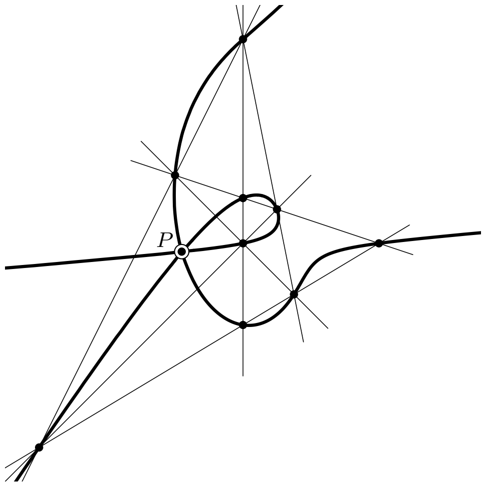


Figure 5: A visualization of an unexpected quartic

Cook II, Harbourne, Migliore and Nagel realized in [11] that the above example is a manifestation of a much more general phenomenon. Their ground-breaking work [11] opened door to an intriguing new theory. In this section I present some of my contributions to its developments.

The authors of [11] coined the term of *unexpected hypersurface* so let me begin with its definition.

Definition 20 (Unexpected hypersurface) *Let $W \subseteq |H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))|$ be a linear system and let $P \in \mathbb{P}^N$ be a general point. We say that for an integer $m \geqslant 1$ W contains an unexpected hypersurface if*

$$\dim(W \otimes I(P)^m) > \max\left\{\dim W - \binom{N+m-1}{N}, -1\right\}.$$

It is elementary to see that if $m = 1$, then there is no unexpected hypersurfaces.

In what follows we restrict our attention to linear systems arising by assigning base points, $W = |H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z))|$, as in the first example described above. Thus the existence of unexpected hypersurfaces becomes a property of the set of assigned base locus Z.

Definition 21 (Sets of points admitting unexpected hypersurfaces) *Let $Z \subseteq \mathbb{P}^N$ be a non-empty finite set of points in $\mathbb{P}^N$. We say that Z admits an unexpected hypersurface of degree d and multiplicity m, if for a general point $P \in \mathbb{P}^N$ there is*

$$h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z) \otimes I(P)^m) > \max\left\{0, h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z)) - \binom{N+m-1}{N}\right\}. \tag{12}$$

It is important to stress here that it is irrelevant if Z itself imposes independent conditions on forms of degree d, or not. In fact, it is easy to see that there exists always a subset $Z' \subseteq Z$ which consists of points imposing independent conditions. Usually such a subset is not uniquely determined.

There are at least two natural generalizations of Definition 21. First of all there is no need to restrict attention to imposed base loci consisting of points. One can take Z to be an arbitrary subscheme of $\mathbb{P}^N$

and adapt condition (12) verbatim with Z a scheme. Some results in this direction are already available in case Z is the union of general lines, see [8], [17].

Another variant, put forward in my article [Hab1], allows a higher number of general points with assigned multiplicities.

Definition 21' *Let $Z \subseteq \mathbb{P}^N$ be a non-empty finite set of points in $\mathbb{P}^N$. We say that Z admits an unexpected hypersurface of degree d and multiplicities $m_1, \ldots, m_r$, if for general points $P_1, \ldots, P_r \in \mathbb{P}^N$ there is*

$$h^0\left(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z) \otimes I(P_1)^{m_1} \ldots \otimes I(P_r)^{m_r}\right) > \max\left\{0, h^0\left(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z)\right) - \binom{N+m_1-1}{N} - \ldots - \binom{N+m_r-1}{N}\right\}.$$

Remark 22 *In this notes we restrict our attention to cases when the right hand side in inequalities in Definitions 21 and 21' is equal to 0. This justifies calling* unexpected *all hypersurfaces defined by global sections of sheaves on the left hand side in inequalities in these Definitions. Typically there is just one such a hypersurface.*

4.3.1 Points dual to Fermat arrangements of lines

I realized in [Hab4] that the points in (10) are dual to the arrangement of lines in $\mathbb{P}^2$ determined by linear factors of the polynomial

$$F_{2,2,2} = xyz(x^2 - y^2)(x^2 - z^2)(y^2 - z^2),$$

where the notation is taken from [Hab4, Example 3.5]. Section 4 in [Hab4] generalizes this example in the following way.

Theorem 23 ([Hab4, Theorem 4.4, Theorem 4.5, Theorem 4.6]) *Let Z be the set of points dual to one of the following polynomials*

- $F_{2,3,1} = xy(x^3 - y^3)(x^3 - z^3)(y^3 - z^3)$, $n = 3$,
- $F_{2,4,0} = x(x^4 - y^4)(x^4 - z^4)(y^4 - z^4)$, $n = 4$,
- $F_{2,n} = (x^n - y^n)(x^n - z^n)(y^n - z^n)$, $n \geqslant 5$.

Then Z admits an unexpected curve of degree $n + 2$ and multiplicity $m = n + 1$.

The general case $n \geqslant 5$ was proved in [11, Theorem 6.12]. The cases $n = 3$ and $n = 4$ are new. Also in [Hab4] (see the proof of Theorem 4.6) I provided explicit equations of unexpected curves, which generalize equation (11):

$$\begin{aligned} f_{P,n}(x:y:z) = xyz\Bigg[&(n+1)\Big(a(b^n + (-1)^{n+1}c^n)x^{n-1} \\ &+ b(a^n + (-1)^{n+1}c^n)y^{n-1} + c(a^n + (-1)^{n+1}b^n)z^{n-1}\Big) \\ &+ \sum_{i=1}^{n-2}(-1)^{i+1}\binom{n+1}{i+1}\Big(a^{n-i}b^{i+1}x^iy^{n-i-1} + a^{n-i}c^{i+1}x^iz^{n-i-1} + b^{n-i}c^{i+1}y^iz^{n-i-1}\Big)\Bigg] \\ &+ a^{n+1}G_x + b^{n+1}G_y + c^{n+1}G_z, \end{aligned}$$

where

$$G_x = yz(y^n - (-z)^n),\ G_y = zx(z^n - (-x)^n),\ G_z = xy(x^n - (-y)^n).$$

4.3.2 Points cut out by hyperplanes in Fermat arrangements

As already mentioned the ideal $I = I(Z)$ of the set Z of points of intersection of lines defined in (5) provided the first non-containment example $I^{(3)} \not\subset I^2$. The set Z admits also an unexpected curve of degree 5 and multiplicity 4. I generalized this observation in [Hab1, Theorem 5] for singular loci of polynomials $F_{2,n}$ with $n \geqslant 3$ and provided explicit equations of unexpected curves.

Theorem 24 (Unexpected curves with a point of multiplicity 4, [Hab1, Theorem 5]) *Let Z be the configuration of points in $\mathbb{P}^2$ consisting of intersection points of lines determined by linear factors of $F_{2,n}$ for $n \geqslant 3$. Let $P = (a : b : c)$ be a general point in $\mathbb{P}^2$. We define the following numbers:*

$$u = \binom{n}{2} - 1, \quad v = \binom{n-1}{2}, \quad w = \binom{n+1}{2}.$$

Then the polynomial

$$\begin{aligned} Q_P(x:y:z) &= -cxy((ub^n + vc^n)(z^n - x^n) + (ua^n + vc^n)(y^n - z^n)) \\ &\quad -bxz((ua^n + vb^n)(y^n - z^n) + (uc^n + vb^n)(x^n - y^n)) \\ &\quad -ayz((ub^n + va^n)(z^n - x^n) + (uc^n + va^n)(x^n - y^n)) \\ &\quad +wa^{n-1}bcx^2(y^n - z^n) + wab^{n-1}cy^2(z^n - x^n) \\ &\quad +wabc^{n-1}z^2(x^n - y^n) \end{aligned}$$

defines the unexpected curve of degree $n + 2$ and multiplicity 4 at P.

A very surprising feature of the series of examples given by Theorem 24 is that while the degree of unexpected curves grows, the multiplicity at the general point remains fixed. It is worth to remember that the foundational paper [11] studies only unexpected curves with degree one more than the multiplicity at the general point, as it happens in the first example in [14].

In [11] the authors restrict their attention to unexpected curves in $\mathbb{P}^2$. The first example of an unexpected hypersurface of higher dimension has been found in [Hab2]. Its construction is based on intersection points of planes defined by linear factors of the polynomial $F_{3,3}$. More precisely we consider the set Z of points, which lie on at least 6 arrangement planes. There are 31 points in Z, the 4 coordinate points and 27 points forming a complete intersection

$$I(Z) = (x^3 - y^3, y^3 - z^3, z^3 - w^3) \cap (x, y, z) \cap (x, y, w) \cap (x, z, w) \cap (y, z, w).$$

The ideal $I(Z)$ turns out to be a binomial ideal and its generators exhibit certain symmetry with respect to the involved variables.

Lemma 25 ([Hab2, Lemma 1]) *The ideal $I(Z)$ is generated by the following 8 binomials of degree 4:*

$$x(y^3 - z^3),\ x(z^3 - w^3),\ y(x^3 - z^3),\ y(z^3 - w^3)\,,$$
$$z(x^3 - y^3),\ z(y^3 - w^3),\ w(x^3 - y^3),\ w(y^3 - z^3)\,.$$

The explicit form of generators is crucial for proving the main result of [Hab2].

Theorem 26 (Unexpected quartic surface in $\mathbb{P}^3$, [Hab2, Theorem 1]) *The set Z defined above admits an unexpected quartic surface S_P with singularity of multiplicity 3 at general point $P = (a : b : c : d)$. The surface S_P is the locus of zeroes of the following polynomial*

$$\begin{aligned} f = f\left((x:y:z:w),(a:b:c:d)\right) &= b^2(c^3 - d^3) \cdot x^3y + a^2(d^3 - c^3) \cdot xy^3 + c^2(d^3 - b^3) \cdot x^3z \\ &+ c^2(a^3 - d^3) \cdot y^3z + a^2(b^3 - d^3) \cdot xz^3 + b^2(d^3 - a^3) \cdot yz^3 \\ &+ d^2(b^3 - c^3) \cdot x^3w + d^2(c^3 - a^3) \cdot y^3w + d^2(a^3 - b^3) \cdot z^3w \\ &+ a^2(c^3 - b^3) \cdot xw^3 + b^2(a^3 - c^3) \cdot yw^3 + c^2(b^3 - a^3) \cdot zw^3. \end{aligned}$$

We remark for the future reference that f can be written as

$$\begin{array}{rcl} f & = & -a^2(b^3-d^3)\cdot x(y^3-z^3)+a^2(b^3-c^3)\cdot x(y^3-w^3) \\ & & +b^2(c^3-a^3)\cdot y(z^3-w^3)-b^2(c^3-d^3)\cdot y(z^3-x^3) \\ & & -c^2(d^3-b^3)\cdot z(w^3-x^3)+c^2(d^3-a^3)\cdot z(w^3-y^3) \\ & & +d^2(a^3-c^3)\cdot w(x^3-y^3)-d^2(a^3-b^3)\cdot w(x^3-z^3). \end{array} \tag{13}$$

For the proof see also [Hab1, Theorem 7].

4.3.3 Unexpected hypersurfaces with multiple general fat points

In [Hab1] I consider a higher dimensional generalization of construction outlined in the section 4.3.2. This study was motivated by the question if sets of points described in Definition 21' exist for $r \geqslant 2$. The main result of [Hab1] is the positive answer to this question.

Turning to the details let Z_N be the set of points in $\mathbb{P}^N$, the union of the complete intersection defined by the ideal

$$(x_0^3 - x_1^3, x_0^3 - x_2^3, \ldots, x_0^3 - x_N^3)$$

together with the $(N+1)$ coordinate points.

The following Lemma generalizes Lemma 25. For the proof see [Hab1, Lemma 6].

Lemma 27 ([Hab1, Lemma 6]) *The ideal $I(Z_N)$ is generated in a single degree 4 by the forms*

$$x_i(x_{i+1}^3 - x_j^3)$$

for $i \in \{0, \ldots, N\}$ and $j \in \{0, \ldots, N\} \setminus \{i, i+1\}$.

Imposing vanishing conditions on elements in $[I(Z_N)]_4$ leads to the first example of an unexpected hypersurface with 2 general fat points.

Theorem 28 ([Hab1, Theorem 10]) *The set Z_N defined above for $N = 5$ admits in $\mathbb{P}^5$ an unexpected hypersurface of degree 4 with multiplicity 3 at general point $P_1 = (a_0 : \ldots : a_5)$ and multiplicity 2 at general point $P_2 = (b_0 : \ldots : b_5)$.*

The proof of Theorem 28 is to some extent geometric and it builds upon the conification method introduced by me and used successfully for the containment problem as well. The idea is to consider cones Ω_{ij} in $\mathbb{P}^5$ (with vertices being 1-dimension flats, more precisely lines L_{ij} joining two coordinate points E_i, E_j, where $E_k = V(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_5)$) over the unexpected quartic surfaces in $\mathbb{P}^3$ whose existence was proved in Theorem 26. These quartic surfaces are taken with respect to appropriate projections of the point P_1. The cones generate a linear system of quartics in $\mathbb{P}^5$, all vanishing at P_1 to order at least 3. It turns out that it is sufficient to consider cones with vertices at lines $L_{01}, L_{02}, L_{03}, L_{12}, L_{13}$ and L_{23}.

Let $\omega_{ij}(x_0 : \ldots : x_5)$ be the equation of Ω_{ij}. Thus ω_{ij} depends on variables with indexes from $\{0, \ldots, 5\} \setminus \{i, j\}$ and the same coordinates of point P_1. For example

$$\omega_{01}(x_0 : \ldots : x_5) = f((x_2 : x_3 : x_4 : x_5), (a_2 : a_3 : a_4 : a_5)),$$

where f is as in Theorem 26. Then the unique unexpected quartic from Theorem 28 is given by the equation

$$\begin{aligned} &\omega_{2,3}(b_0 : \ldots : b_5)\cdot\omega_{0,1}(x_0 : \ldots : x_5) - \omega_{1,3}(b_0 : \ldots : b_5)\cdot\omega_{0,2}(x_0 : \ldots : x_5) \\ &+\omega_{0,3}(b_0 : \ldots : b_5)\cdot\omega_{1,2}(x_0 : \ldots : x_5) + \omega_{1,2}(b_0 : \ldots : b_5)\cdot\omega_{0,3}(x_0 : \ldots : x_5) \\ &-\omega_{0,2}(b_0 : \ldots : b_5)\cdot\omega_{1,3}(x_0 : \ldots : x_5) + \omega_{0,1}(b_0 : \ldots : b_5)\cdot\omega_{2,3}(x_0 : \ldots : x_5). \end{aligned}$$

Note that the dependence on the coordinates of the point P_1 is hidden in the polynomials $\omega_{i,j}$.

I expect that the construction from Theorem 28 in $\mathbb{P}^5$ extends to projective spaces of arbitrary high odd dimension.

Conjecture 29 (see [Hab1, Section 5]) *For $k \geqslant 1$ let $N = 2k+1$. The set $Z_N \subseteq \mathbb{P}^N$ admits an unexpected quartic vanishing to order 3 at a general point $P_1 = (a_0 : \ldots : a_N)$ and to order 2 at general points $P_i = (b_0^i : \ldots : b_N^i)$ for $i = 2, \ldots, k$.*

This conjecture is proved for $k = 1$ in [Hab2], $k = 2$ in [Hab1] and verified by computer for $k = 3$ and $k = 4$. Moreover, I have a conjectural recursive explicit equation ω_N for the unexpected quartic in $\mathbb{P}^N$ which again builds upon the conification method

$$\omega_N(x) = \sum_{i=0}^{N} a_i^2 \sum_{j=i+2}^{N+i} K(i,j) \cdot \omega_{N-2}^{i,j}(b_0^k : \ldots : b_N^k) \cdot x_i \left(x^3_{(i+1 \mod (N+1))} - x^3_{(j \mod (N+1))} \right),$$

where $K(i,j) = \operatorname{sgn}(j-i) \cdot (-1)^{j+i}$ is the sign function and $\omega_{N-2}^{i,j}$ is the unexpected quartic in variables $x_0, \ldots, x_N$ with x_i, $x_{(j \mod (N+1))}$ omitted.

4.3.4 BMSS duality

In the work of Di Gennaro, Ilardi and Vallés [14] the existence of the unexpected curve follows from the non-vanishing of certain cohomology group. The authors made no effort to write down its equation explicitly. Similarly, the study of Cook II, Harbourne, Migliore and Nagel in [11] is of theoretical nature. In [Hab2] we provided explicit equation of the unexpected quartic curve admitted by the B_3 root system (11) and also of the unexpected quartic surface (13) admitted by a Fermat-type configuration of points in $\mathbb{P}^3$. These results provided additional insights and opened a new path of research. The idea is very simple, the equation of an unexpected hypersurface, e.g. of the unexpected quartic curve in (11) or surface in (13), can be viewed as a polynomial in variables, which are coordinates of the general point. For example (11) can be consider as a polynomial in variables $(a : b : c)$. In general, we obtain a bihomogeneous polynomial

$$f((a_0 : \ldots : a_N), (x_0 : \ldots : x_N)), \tag{14}$$

where $P = (a_0 : \ldots : a_N)$ is the general point.

This simple observation made in [Hab2] has considerably more general implications. To begin with, if f defines an unexpected hypersurface of degree d and multiplicity m at P, then it has degree d in the second set of variables and degree greater or equal m in the first set of variables. A considerable part of work of Harbourne, Migliore, Nagel and Teitler in [25], which is a natural extension of [11] is motivated by the above observation. In particular, the authors introduced the notion of the BMSS duality (the name honors the authors of [Hab2] comprising our initials). Research in this direction culminated in the following theorem, cf. [25, Theorem 4.4].

Theorem 30 *Let $f(a,x) = f((a_0 : a_1 : a_2), (x_0 : x_1 : x_2))$ be a bihomogeneous polynomial of bi-degree $(m, m+1)$ such that*

- *for a general point $P = (a_0 : a_1 : a_2)$, the polynomial $f(P, (x_0 : x_1 : x_2))$ is reduced and irreducible and;*
- *for fixed $P = (a_0 : a_1 : a_2)$, f vanishes to order m at P (as a polynomial in variables $(x_0 : x_1 : x_2)$) and;*
- *for fixed $Q = (x_0 : x_1 : x_2)$, f vanishes to order m at Q (as a polynomial in variables $(a_0 : a_1 : a_2)$).*

Then $f(a,x)$ is the equation of the tangent cone at $x = P$ of the curve $\{f(P,x) = 0\}$.

Our recent work [16] continues the study of the BMSS duality from another point of view, namely the duality is related to properties of suitable interpolation matrix.

Example 31 (see [Hab2, Section 2]) *In the specific situation of the unexpected curve* (11) *Theorem 30 amounts to identifying the set*

$$\{(x:y:z)\in\mathbb{P}^2 : 3x(y^2-z^2)\cdot a^2bc + 3y(z^2-x^2)\cdot ab^2c + 3z(x^2-y^2)\cdot abc^2 \\ + x^3\cdot b^3c - x^3\cdot bc^3 + y^3\cdot ac^3 - y^3\cdot a^3c + z^3\cdot a^3b - z^3\cdot ab^3 = 0\}$$

as the tangent cone to the curve

$$\{3a(b^2-c^2)\cdot x^2yz + 3b(c^2-a^2)\cdot xy^2z + 3c(a^2-b^2)\cdot xyz^2 \\ + a^3\cdot y^3z - a^3\cdot yz^3 + b^3\cdot xz^3 - b^3\cdot x^3z + c^3\cdot x^3y - c^3\cdot xy^3 = 0\}$$

at $P=(a:b:c)$*.*

It is worth to remark that Theorem 30 is far from being satisfactory since it handles only plane curves under a rather restrictive assumption that their degree and multiplicity at the general point differ by 1. It is expected that BMSS duality explains the geometry of the tangent cone of an unexpected hypersurface at a general point in much more general setting. Some steps towards establishing this kind of connections have been undertaken in [16]. Additional evidence and further discussion is provided by examples studied in our recent work [15]. However a complete solution to this problem remains to be established. It is a challenging path of further research.

4.3.5 Osculating spaces and companion varieties

Unexpected hypersurfaces are related to some differential geometric properties of Veronese-type maps on projective spaces. We begin by recalling a classical construction.

Let X be a smooth, complete, complex variety of dimension n and let L be a line bundle on X. Let $V=H^0(X,L)$. The m-jet bundle of L is the coherent sheaf

$$\mathcal{J}_m(L) = (p_1)_*(p_2^*L\otimes \mathcal{O}_{X\times X}/\mathcal{I}(\Delta)^{m+1}),$$

where $\Delta\subseteq X\times X$ is the diagonal and p_1, p_2 are the projection maps

$$\begin{array}{ccc} & X\times X & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X & & X \end{array}.$$

This sheaf is locally free of rank $\binom{n+m}{n}$ and its fiber at a point P can be identified with

$$\mathcal{J}_m(L)_P \simeq H^0(X, L\otimes\mathcal{O}_X/\mathcal{I}(P)^{m+1}),$$

where $\mathcal{I}(P)$ is the ideal sheaf of $P\in X$.

Definition 32 (The m-th osculating space) *The m-th osculating space* $\mathrm{Osc}_P^{(m)}(X)$ *at* $P\in X$ *is the projectivization* $\mathbb{P}(j_{k,P}(V))\subseteq\mathbb{P}(V)$ *of the image of the evaluation map*

$$j_{k,P}: V\longrightarrow H^0(X,L\otimes\mathcal{O}_X/\mathcal{I}(P)^{m+1}).$$

It is expected that the space $\mathrm{Osc}_P^{(m)}(X)$ at a general point $P\in X$ has the (projective) dimension $\binom{n+m}{n}-1$. If this dimension is lower than $\binom{n+m}{n}-1$ at every point, then following Shifrin [40] we say that X is *hypo-osculating of order m*. More precisely, the deficiency of osculating spaces of X is measured by the number of Laplace equations X satisfies.

Definition 33 (Laplace equation) *A projective variety $X \subset \mathbb{P}^N$ of dimension n satisfies δ independent Laplace equations of order m, if for a general point $P \in X$ $\dim \operatorname{Osc}_P^{(m)} X < \min \left\{ \binom{n+m}{n} - 1, N \right\}$ and there is*

$$\delta = \min \left\{ \binom{n+m}{n} - 1, N \right\} - \dim \operatorname{Osc}_P^{(m)} X.$$

The existence of unexpected hypersurfaces of degree d admitted by a set Z in $\mathbb{P}^N$ is reflected by the existence of osculating spaces of dimension less than expected of the image of $\mathbb{P}^N$ under the rational map defined by the linear system of homogeneous polynomials of degree d in $\mathbb{C}[x_0, \ldots, x_N]$ vanishing at all points of Z.

More precisely, let $Z \subseteq \mathbb{P}^N$ be a set of points admitting an unexpected hypersurface of degree d with multiplicity m at a general point. Let $V \subseteq H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$ be the space of sections vanishing at Z, i.e., $V = H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes \mathcal{I}(Z))$. Elements of V are in one-to-one correspondence to the graded part $(I(Z))_d$ of degree d of the ideal $I(Z)$ of Z. They determine a rational map

$$\varphi_V : \mathbb{P}^N \dashrightarrow \mathbb{P}^M,$$

where $M = \dim(V) - 1$. The indeterminancy locus Z of φ_V is easily resolved by passing to the blow up $\pi : Y \to \mathbb{P}^N$ of Z. We denote the resulting morphism by $\widetilde{\varphi}_V$

$$\begin{array}{ccc} Y & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\widetilde{\varphi}_V} & \\ \mathbb{P}^N & \overset{\varphi_V}{\dashrightarrow} & \mathbb{P}^M. \end{array}$$

Lemma 34 *Let $X = \widetilde{\varphi}_V(Y)$ be the closure of the image of φ_V. Then X satisfies at least one Laplace equation of order $(m-1)$.*

This is easily seen as follows. The fact that there exists an unexpected hypersurface H_P of degree d with a point of multiplicity m at P is equivalent to the expected dimension of $\operatorname{Osc}^{(m)}_{\widetilde{\varphi}_V(P)} X$ being $\geqslant M$, whereas the unexpected hypersurface H_P corresponds to an element in V, hence in $\mathcal{O}_{\mathbb{P}^M}(1)$, cutting out on X a divisor D_P with $\operatorname{mult}_{\widetilde{\varphi}_V(P)} D_P \geqslant m$.

Combining above ideas with the BSMM duality leads to the concept of *companion varieties*, which I introduced in [Hab3]. The idea is in principle very simple. Let Z be a set of points in $\mathbb{P}^N$ admitting an unexpected hypersurface of degree d and multiplicity m at a general point $P = (a_0 : \ldots : a_N)$. Let $g_0, \ldots, g_M$ be generators of $[I(Z)]_d$. The equation of the unexpected hypersurface (14) can be written as

$$F((a_0 : \ldots : a_N), (x_0 : \ldots : x_N)) = \sum_{i=0}^{M} h_i(a_0 : \ldots : a_N) g_i(x_0 : \ldots : x_N).$$

Definition 35 (Companion variety, see [Hab3, Subsection 4.4]) *In the above setting let X be the image of the rational map determined by $g_0, \ldots, g_M$. The companion variety X' of X is then the image of the rational map determined by $h_0, \ldots, h_M$.*

It is natural to wonder what geometric properties distinguish companion varieties among all rational varieties, or more precisely projections of Veronese varieties. In [Hab3] I showed the following result.

Theorem 36 (see [Hab3], Proposition 4.12) *Let Z be the set of points in $\mathbb{P}^2$ determined by the B_3 root system. Then, the companion variety X' determined by unexpected quartics admitted by Z is smooth surface of degree 9 in $\mathbb{P}^5$ satisfying one Laplace equation of order 2.*

It was also observed in [Hab3] that in case of the B_3 root system the algebra $\mathbb{C}[x, y, z]/\langle g_0, g_1, \ldots, g_5 \rangle$ is artinian. This implies, in particular, that the rational map determined by $g_0, \ldots, g_5$ is a morphism.

These results are very surprising and have triggered further research. New examples and phenomena have appeared in the recent preprint [15], which I co-authored. I expect that a number other contributions will follow.

4.3.6 Summary

This section highlights my contributions presented in the achievement.

1. Introduction and systematic exploration of Fermat-type arrangements of hyperplanes in projective spaces of arbitrary dimension. This idea is present in [Hab1], [Hab2], [Hab4], [Hab5] and [Hab6].
2. Relating the existence of unexpected hypersurfaces to higher osculating spaces in [Hab3] and introduction of companion varieties.
3. Initiation of systematic study of parameter spaces of Böröczky arrangements in [Hab7]. The algebraic method is in a sense universal and can be applied to other types of arrangements.
4. Introduction and application of the conification to some problems of algebraic character. It is used in [Hab1], [Hab5] and [Hab6], which shows that it is applicable in a wide range of problems.

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4.4 Paths of research not included in the achievement

In this section I comment briefly my works not included in the achievement. They are presented in groups. The grouping is subjective. The main objective was to keep this presentation short.

A. Asymptotic invariants of homogeneous ideals

In recent years it has become apparent that adopting an asymptotic perspective in commutative algebra and algebraic geometry often leads to more regular and clear results than those obtained considering isolated phenomena. In this spirit a number of asymptotic invariants has appeared and been studied recently. Some of them have deep roots in mathematics. For example, the initial degree $\alpha(I)$ of a homogeneous ideal I is defined as the least degree of a non-zero element in I. The asymptotic counterpart of $\alpha(I)$ is the Waldschmidt constant $\widehat{\alpha}(I)$, which takes into account all symbolic powers of I simultaneously

$$\widehat{\alpha}(I) = \inf_{m \geqslant 1} \frac{\alpha(I^{(m)})}{m}. \tag{15}$$

It turns out that this invariant defined as in (15) around 2010 was studied, well before, in complex analysis in connection with higher dimensional variants of the Schwarz Lemma. More specifically, Moreau proved the following version of the Schwarz Lemma in several complex variables, [33, Theorem 1.1].

Theorem 37 (Moreau) *Let $Z \subset \mathbb{C}^n$ be a finite set of points. For every positive $m \in \mathbb{Z}$, there exists a real number $r_m(Z)$ such that for all $R \geqslant r \geqslant r_m(Z)$ and for all holomorphic functions f vanishing to order at least m at every point of Z there is*

$$|f|_r \leqslant \left(\frac{2e^{n/2}r}{R} \right)^{\alpha(mZ)} |f|_R, \tag{16}$$

where $|f|_s = \sup_{|z| \leqslant s} |f(z)|$ and $\alpha(kW)$ is the least degree of a polynomial vanishing at all points of a finite set W to order at least k.

We study Waldschmidt constants from various points of view in [A1, A2, A3, A4]. Our work [A7] is of slightly different flavor. We consider the difference of values of the initial degree of the second symbolic power and of the ideal itself. Small value of this difference has strong impact on the geometry of the underlying set of points.

The idea to attach objects of convex geometry to polynomials is present in algebraic geometry for a long time in form of Newton polytopes. In the realms of linear systems it has been generalized considerably in works of Okounkov around 1995 and systematically applied to important questions in positivity theory by Lazarsfeld, Mustaţă and Kaveh, Khovanskii around 2006. The general idea behind these construction is that a geometrical object encodes simultaneously various single numerical invariants, e.g. Waldschmidt constants. Our paper [A8] explores some properties of Newton-Okounkov bodies. Article [A5], [A6] deal in turn with related concept of the limiting shape attached to a graded family of ideals. They show in particular how to read certain important properties of such families: Waldschmidt constants, asymptotic regularity and asymptotic Hilbert functions out of the geometry of the attached limiting shape.

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B. Combinatorial aspects of algebraic geometry

Line arrangements are a classical topic in geometry. In articles collected here our interest focuses on some combinatorial aspects of this area. My articles [B2] and [B3] provide introduction to this circle of ideas. Even though the problems are approached partly by combinatorial methods (most notably in [B1]) we address questions rooted in algebra and algebraic geometry.

One of these problems is motivated by Bounded Negativity Conjecture which predicts that on any fixed smooth complex projective surface there is a number bounding from below the selfintersection of irreducible curves contained in the surface. This is not verified even in the case of rational surfaces. Special attention is given to surfaces arising as blow ups of $\mathbb{P}^2$ in intersection points of line arrangements. The linear Harbourne constants, studied in [B4] and [B7] focuses on such surfaces.

Article [B6] in turn is motivated by prominent role of line arrangements with many intersection points of multiplicity at least 3 in constructing non-containment examples for the $I^{(3)} \subseteq I^2$ problem. We provide a full classification of arrangements of at most 11 lines with maximal possible number of triple points. We discuss their realizability in dependence of the ground field. Our results complete or improve some earlier results in this direction.

Finally, our article [B5] is motivated by celebrated Sylvester-Gallai Theorem which shows, in particular, that there is no line arrangement in the *real* projective plane, where through any intersection point of two lines, there is another line passing (excluding the pencil case). We generalize this kind of statement to arrangements of real conics and we rise a question if the restriction to the real projective plane is important in this case. Our results have been generalized to curves of degree 3 by Alex Cohen and Frank de Zeeuw in "A Sylvester-Gallai theorem for cubic curves" (arXiv:2010.01513).

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[B2] **Szpond, J.**: A few introductory remarks on line arrangements, Analytic and Algebraic Geometry 3, Łódź University Press 2019, 201 – 212

[B3] **Szpond, J.**: On Hirzebruch type inequalities and applications, In: Alberich-Carraminana M., Galindo C., Küronya A., Roé J. (eds) Extended Abstracts February 2016. Trends in Mathematics, vol 9. pp. 89 – 94, Birkhäuser, Cham 2018

[B4] Dumnicki, M., Harrer, D., **Szpond, J.**: On absolute linear Harbourne constants, Finite Fields Appl. 51 (2018), 371 – 387

[B5] Czapliński, A., Dumnicki, M., Farnik, Ł., Gwoździewicz, J., Lampa-Baczyńska, M., Malara, G., Szemberg, T., **Szpond, J.**, Tutaj-Gasińska, H.: On the Sylvester-Gallai theorem for conics, Rend. Sem. Mat. Univ. Padova 136 (2016), 191 – 203

[B6] Dumnicki, M., Farnik, Ł., Główka, A., Lampa-Baczyńska, M., Malara, G., Szemberg, T., **Szpond, J.**, Tutaj-Gasińska, H.: Line arrangements with the maximal number of triple points, Geom. Dedicata, 180 (2016), 69 – 83

[B7] **Szpond, J.**: On linear Harbourne constants, British Journal of Mathematics & Computer Science, 8(4) (2015), 286 – 297

C. Linear systems and positivity questions

Results concerning linear series are most scattered. On the one hand side, the paper [C3] is motivated by the Bounded Negativity Conjecture and studies negative curves on blow ups of $\mathbb{P}^2$ in certain symmetric set of points. The main result of [C4] is the following theorem which provides sharp constrains on possible values of Seshadri constants of ample line bundle on surfaces. It is well known that if L is an ample line bundle on a smooth complex surface X, then its Seshadri constant $\varepsilon(L;1)$ is either maximal possible, i.e., equal to $\sqrt{L^2}$ or it is smaller than this number and rational. A series of results by Szemberg relates possible values of Seshadri constants $\varepsilon(L;1)$ computed in very general point x to solutions of Pell's equation

$$y^2 - dx^2 = 1, \tag{17}$$

where $d = L^2$. Our Theorem provides additional restrictions on $\varepsilon(L;1)$ derived from a series of works of Küronya and Lozovanu.

Theorem 38 *Let X be a smooth projective surface, $x \in X$, L an ample line bundle on X such that $(L^2) = d$ is not a square. Let (p, q) be an arbitrary solution of the Pell equation* (17). *Then either*

$$\varepsilon(L;1) \geqslant \frac{p}{q}d,$$

or

$$\varepsilon(L;1) \in \left\{1, 2, \ldots, \lfloor\sqrt{d}\rfloor\right\} \cup \left\{\frac{a}{b} \;\; \text{such that} \;\; 1 \leqslant \frac{a}{b} < \frac{p}{q}d \;\; \text{and} \;\; 2 \leqslant b < q^2\right\}.$$

Article [C1] and [C2] revolve around the postulation problems. In [C2] we generalize an old result of Hartshorne and Hirschowitz to the effect that a finite set of s general lines in $\mathbb{P}^n$ has good postulation, i.e., they imposes the expected number $s(d+1)$ of conditions on forms of degree d. We show that this holds if one allows one of the lines to have a non-reduced structure. The paper introduces zig-zags as convenient degenerations of general lines, which makes Castelnuovo idea with the trace and residual schemes in specialization problem work.

Article [C1] in turn studies postulation of sets of points, with one point with non-reduced structure. As explained in Section 4.3 in this case failures to good postulation are possible. We study them from the perspective of the interpolation matrix.

[C1] Dumnicki, M., Farnik, Ł., Harbourne, B., Malara, G., **Szpond, J.**, Tutaj-Gasińska, H.: A matrixwise approach to unexpected hypersurfaces, Lin. Alg. and Appl. 592 (2020), 113 – 133

[C2] Bauer, T., Di Rocco, S., Schmitz, D., Szemberg, T., **Szpond, J.**: On the postulation of lines and a fat line, J. Symbolic Comput. 91 (2019), 3 – 16

[C3] Dumnicki, M., Farnik, Ł., Hanumanthu, K., Malara, G., Szemberg, T., **Szpond, J.**, Tutaj-Gasińska, H.: Negative curves on special rational surfaces, Analytic and Algebraic Geometry 3, Łódź University Press 2019, 6 – 78

[C4] Farnik, Ł., Szemberg, T., **Szpond, J.**, Tutaj-Gasińska, H.: Restrictions on Seshadri Constants on Surfaces, Taiwanese J. Math. 21 (2017), 27 – 41

D. Symbolic powers and other geometrically motivated operations on homogeneous ideals

The last group of articles co-authored by me revolves around symbolic powers of homogeneous ideals and their relation to ordinary powers. Article [D4] contains the first non-containment example for the $I^{(3)} \subset I^2$ problem of Huneke defined over the real numbers. It builds upon Böröczky arrangements. Joint work [D2] with Malara provides a rational, simplicial non-containment example. The survey [D3] with Szemberg provides the up to date overview of the containment problem. It contains problems and suggestions which stimulated research in this direction in the next years. In particular, the approach taken on by Grifo in her recent works, has some roots in our paper. The article [D1] resulted from a workshop in Oberwolfach in 2015 devoted to symbolic powers. In this paper we are interested in the natural question when symbolic and ordinary powers are equal. This is always the case for complete intersection ideals but there are many other cases. Our main result, presented below, provides nice, effective criteria for verifying if symbolic and ordinary powers agree for codimension two Cohen-Macaulay ideals.

Theorem 39 *Let $I = I_X$ be the saturated homogeneous ideal defining a subscheme $X \subset \mathbb{P}^n$ such that* $\operatorname{codim}(X) = 2$*, X is arithmetically Cohen-Macaulay and X is locally a complete intersection. Then the following conditions are equivalent:*

(a) $I^{(n)} = I^n$;

(b) $I^{(m)} = I^m$ *for all* $m \geqslant 1$;

(c) *I has at most n minimal generators.*

The statement is interesting, because for schemes X as in the Theorem $I_X^{(m)} = I_X^m$ for $m < n$. Thus it is exactly the n-th power which decides about the equality of all powers. In the paper we address also codimension 3 subschemes under the assumption that they are arithmetically Gorenstein.

[D1] Cooper, S., Fatabbi, G., Guardo, E., Lorenzini, A., Migliore, J., Nagel, U., Seceleanu, A., **Szpond, J.**, Van Tuyl, A.: Symbolic powers of codimension two Cohen-Macaulay ideals, Communications in Algebra, vol. 48 (11) (2020), 4663 – 4680

[D2] Malara, G., **Szpond, J.**: The containment problem and a rational simplicial arrangement, Electron. Res. Announc. Math. Sci. 24 (2017), 123 – 128

[D3] Szemberg, T., **Szpond, J.**: On the containment problem, J. Rend. Circ. Mat. Palermo, II. Ser 66 (2017), no. 2, 233 – 245

[D4] Czapliński, A., Główka, A., Malara, G., Lampa-Baczyńska, M., Łuszcz-Świdecka, P., Pokora, P., **Szpond, J.**: A counterexample to the containment $I^{(3)} \subseteq I^2$ over the reals, Adv. Geometry, 16 (2016), 77 – 82

5 Presentation of significant scientific or artistic activity carried out at more than one university, scientific or cultural institution, especially at foreign institutions

- Germany, Oberwolfach Research Fellow, September, 2020,
- USA, University of Nebrasca-Lincoln, September, 2019,
- Germany, University of Marburg, March, 2018,
- Sweden, KTH Stockholm, November, 2017,
- Sweden, KTH Stockholm, November, 2016,
- Poland, Institute of Mathematics, Polish Academy of Science, April – June, 2016,
- Italy, University of Catania, November, 2015,
- Germany, University of Freiburg, June, 2015.

6 Presentation of teaching and organizational achievements as well as achievements in popularization of science or art

Teaching:

- Courses taught: Geometry, Differential Geometry (in English for Erasmus students), Linear Algebra, Commutative Algebra, Topology, Analysis, Numerical Methods, Discrete Mathematics, Effective Methods in Algebraic Geometry, Effective Methods in Discrete Mathematics (for PhD students).
- Author of handbook for students:
  J. Szczawińska, **J. Szpond**, *Geometria elementarna. Notatki do wykładu* (Elementary Geometry. Notes for students), Wydawnictwo Szkolne OMEGA, Kraków, 3 editions: 2016, 2017, 2019.
- Advisor of diploma thesis: 23 bachelor, 16 master.

Organizational achievements:

- Chair of Geometry, in Institute of Mathematics, Pedagogical University of Cracow, September 2017 - September 2020.
- Co-organized events:
  - ICM 2022 Satellite Conference "Recent Advances in Classical Algebraic Geometry", Cracow, Poland, June 28 – July 2, 2022,
  - "MEGA 2022 Conference", Cracow, Poland, June 20 - 24, 2022,
  - "Oblicza Algebry IV (Shades of Algebra IV)", Cracow, Poland, May 27 - 30, 2021,
  - "Lefschetz Properties in Algebra, Geometry and Combinatorics", Oberwolfach, Germany September 27 - October 3, 2020,
  - "Oblicza Algebry III (Shades of Algebra III)", Cracow, Poland, June 1 - 4, 2019,
  - "Asymptotic Invariants of Homogeneous Ideals", Oberwolfach, Germany, September 30 - October 6, 2018,
  - The Twentieth Andrzej Jankowski Memorial Lecture Introductory Workshop, Cracow, Poland, April 20 - 22, 2018,

- "Oblicza Algebry II (Shades of Algebra II)", Cracow, Poland, June 1 - 4, 2017,
  - The Nineteenth Andrzej Jankowski Memorial Lecture Introductory Workshop, Cracow, Poland, April 21 - 23, 2017,
  - Coordination of European Researches' Night (MSCA-NIGHT), Cracow, Poland, 2016, 2017,
  - Young Researchers Seminar coordinator – part of Simons Mini-Semester "Polish Algebraic Geometry mini-Semester", April - June, 2016,
  - "Oblicza Algebry (Shades of Algebra)", Cracow, Poland, May 29 - June 1, 2015,
  - IMPANGA 15, Banach Center, Będlewo, Poland, April 12 - 18, 2015.

- Member of the Selection Comittee for doctoral studies at the Pedagogical University of Cracow, 2011 – 2019.

Popularization:

- Author of two snapshots of modern mathematics from Oberwolfach:
  - On the containment problem, No.3/2016, 10.14760/SNAP-2016-003-EN (2016), (with T. Szemberg),
  - A few shades of interpolation, No. 7/2017, 10.14760/SNAP-2017-007-EN (2017).
- Co-investigator of "Laboratory of Creative Mathematics", The National Centre of Research and Development grant Nr POWR.03.01.00-00-C008/16, September 2017 – August 2019.
- Coordinator and co-investigator of "Exploratory of Creative Mathematics", The National Centre of Research and Development grant Nr POWR.03.01.00-00-U126/17-01, September 2018 – August 2020.
- Organizer of Workshop: Are lines straight (forward)? at EuroMath 2018, Cracow, Poland.

7 Apart from information set out in 1-6 above, the applicant may include other information about his/her professional career, which he/she deems important

- Managing Editor of the journal Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica, since January 1, 2017.
- Reviewer for Zentralblatt MATH.
- Referee for various mathematical journals (Iranian Mathematical Society, Annales Polonici Mathematici, Communications in Algebra, Journal of Pure and Applied Algebra, Bulletin of the Malaysian Mathematical Sciences Society, Journal of Number Theory, Comptes Rendus Mathématique, Mathematics, Journal of Algebraic Combinatorics, Collectanea Mathematica).
- Scientific grants:
  - National Science Centre Poland, Opus 18 Grant Nr 2019/35/B/ST1/00723, co–investigator, June 2020 - June 2023,
  - National Science Centre Poland, Harmonia Grant Nr 2018/30/M/ST1/00148, co–investigator, March 2019 - March 2022,
  - National Science Centre Poland, Miniatura 2 Grant nr 2018/02/X/ST1/00519, **principal investigator**, December 2018 - December 2019,
  - National Science Centre Poland, Opus 8 Grant Nr 2014/15/B/ST1/02197, co–investigator, July 2015 – July 2018,

- Solidarity travel grant European Mathematical Society, June 2017.

- Co-advisor of PhD thesis: *The effect of points fattening on del Pezzo surfaces* of Magdalena Lampa-Baczyńska from Pedagogical University of Cracow, 2017.
- Honorably mentioned in the Edyta Szymańska competition for the best paper in mathematics authored by a female mathematician, 2019, 2021.
- Rector's prize for scientific achievements in 2018, 2019.
- Medal of the National Education Commission, 2018.
- Bronze Medal for Long Service, 2016.

Justyna Szpond
(Applicant's signature)