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Doctoral Thesis

Proof systems for some many-valued and modal logics

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Declaration of Authorship

I, Yaroslav Petrukhin, declare that this thesis titled, "Proof systems for some many-valued and modal logics" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: ________________________________

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Contents

Chapter 1

Introduction and motivation

The aim of this thesis is to explore proof systems for various modal, many-valued, and modal many-valued logics. We consider two kinds of proof systems: sequent calculi (and their generalisations: hypersequent and nested sequent calculi) and natural deduction systems. Sequent calculi are known to be a good theoretical tool for the investigation of proofs; they visually and clearly represent the structure of the proofs and allow the distinction between logical and structural rules to distinguish the properties of the logical connectives and the properties of the consequence relation. One of the central theorems of proof theory is the cut elimination theorem, or the cut admissibility theorem. The cut elimination theorem says that if in a sequent calculus with a cut rule as a primitive we are able to prove some sequent, then we are able to prove the same sequent in its cut-free version. The cut admissibility theorem says that if in a cut-free sequent calculus we have proofs of the premisses of a cut rule, then without using the cut rule itself, we are able to get its conclusion. As a consequence of the cut elimination/admissibility theorem, it is often possible to reach the subformula property (or some of its restricted forms), interpolation, decidability, and other important results. The advantage of natural deduction systems is their similarity to the process of natural human reasoning.

In the case of natural deduction systems, there is an analogue of the cut elimination/admissibility theorem: the normalisation theorem. It says that in the proof there are no maximal formulas (the formulas that are conclusions of introduction rules and major premisses of elimination rules) and no maximal segments (i.e., sequences of maximal formulas). In our case, we will use a slightly different form of normalisation since we are going to deal with general elimination and general introduction rules (it is easier to formulate calculi working with arbitrary n-ary Boolean or three- or four-valued connectives in a uniform way with the help of such rules), so for us, a maximal formula is the one that is a major premiss of a general elimination rule and a major assumption of a general introduction rule. Anyway, the normalisation theorem helps establish the subformula property.

During our investigation, we will try to provide for any logic in question both a sequent calculus (or rather, a hypersequent or a nested sequent calculus) and a natural deduction system. We will prove the cut admissibility theorem for all the sequent calculi in question and do that by two methods: a semantic one (as a consequence of a completeness theorem proven by a Hintikka-style argument, the most popular and convenient method of proving completeness theorem for sequent calculi) and a syntactic constructive one (for hypersequent calculi we use the method introduced by Metcalfe, Olivetti, and Gabbay [123] and further developed by Ciabattoni, Metcalfe, and Montagna [26], it seems to be the most effective and powerful method for hypersequent sequent calculi; for nested sequent calculi we adopt Poggiolesi's proof [157], since our nested sequent calculi are modifications of Poggiolesi's ones). The normalisation theorem will be proved by the syntactic constructive method used by Kürbis [104, 103] which is especially useful for the natural deduction systems such that all their rules are general introduction and general elimination rules. Although we consider different logics from two different domains of non-classical logic, we prefer to use the same methodology to study them. So for the logics in question, we present both sequent (hypersequent, nested sequent) calculi and natural deduction systems. For any sequent (hypersequent, nested sequent) calculus in question, we provide a Hintikka-style completeness proof; as a consequence, we get the cut admissibility theorem and the subformula property; then we present a constructive cut admissibility proof. For any natural deduction system in question, we give a Henkin-style completeness proof or a proof of its equivalence to the corresponding sequent calculus (if the completeness theorem for a natural deduction system in question has not been proven before in the literature), a constructive syntactic normalisation proof, and establish the subformula property.

We have said some general words about the proof systems we are going to investigate. Let us say a few words about logics for which these proof systems are developed. We explore two different fields of non-classical logics: modal and many-valued ones, including their mixture, modal many-valued logics, and their algebraic generalisation — modal multilattice logics. In general, logical systems can be divided into two groups: tabular and non-tabular ones. The former have finitely-valued semantics. The latter require infinitely-valued semantics or other types of semantics: Kripke, algebraic, topological, etc. Since three- and four-valued logics are the most popular and remarkable representatives of finitely-valued logics (although one can find papers on five- [166, 27], six- [186, 72, 56, 27], eight- [198, 88], nine- [72], and even sixteen-valued [179] logics), we are going to focus our attention on them. There are lots of various types of non-tabular logics: modal, intuitionistic, linear, temporal, epistemic, doxastic, dynamic, relevant, fuzzy, etc. Since most of them have Kripke semantics, which was originally developed for modal logics, we think that it is reasonable to focus on them. Non-tabular modal logics have not only Kripke semantics, but many-valued ones as well (e.g., **S5** has an infinitely-valued semantics due to Prior [167], **S4** has a semantics based on an infinite sequence of finitely-valued matrices due to Dummett and Lemmon [37] based on Jaśkowski's [86] matrix for intuitionistic logic). Thus, at least some modal logics might be viewed as many-valued ones. So we think that many-valued and modal logics might be called the most representative domains of non-classical logics, and that explains our choice of this topic.

Now we would like to briefly describe the content and motivation of each of the chapters of this work. A more detailed explanation is presented at the beginning of each chapter: all of them have their own introductions.

Chapter 2 is devoted to proof theory for modal logics. It is a well-developed area of research. A lot of works have been written on this topic by various scholars; we refer just to some surveys of them. One may read Indrzejczak's book [77] about natural deduction and sequent calculi for modal logics, Poggiolesi's book [157] about sequent calculi and their generalisations for modal logics, Fitting's book [54] about various proof systems for modal logics, and many more. We would like to emphasise that most of the studies on proof theory for modal logics are devoted to necessity and possibility operators (we call them standard modalities). However, one may find in the literature some other modalities that are philosophically important; we call them non-standard, such as contingency and non-contingency, essence and accidence (these two can be in two versions: the statement is essentially or accidentally *true* and the statement is essentially or accidentally *false*), negated modalities: impossibility and unnecessity (which can also be viewed as paracomplete and paraconsistent negations). There are some works on proof theory for these kinds of modalities (we discuss them in Chapter 2), but, first, there is no systematic consideration of these modalities in the proof-theoretic framework; second, as we will see in Chapter 2, the existing works quite often propose not-cut-free sequent calculi and do not deal with natural deduction. We would like to fill this gap in our dissertation and systematically apply the proof-theoretic methods developed for necessity and possibility to non-standard modalities. It seems that non-standard modalities more often than standard ones require generalisations of ordinary sequent calculi to obtain a cut elimination theorem. So for all of them, we use either hypersequent calculi or even more general nested sequent calculi. Using Restall's [168] hypersequent calculus for **S5** as well as Avron and Lahav's [9] hypersequent calculus for its paraconsistent version (with a paraconsistent negation defined as a classical negation of necessity) offered by Béziau [15], we formulate hypersequent calculi for the logics with non-standard modalities (we call them non-standard modal logics). Using Poggiolesi's [157] nested sequent calculi for **K** and its reflexive, serial, symmetric, and transitive extensions as a starting point, we introduce nested sequent calculi for non-standard modal logics. Then we will switch to natural deduction: first, we give a new proof of normalisation for Segerberg's [173] natural deduction system for classical propositional logic formulated in a language with at least one arbitrary n-ary Boolean connective. Our proof generalises Kürbis' method, while the previously known one by Geuvers and Hurkens [62] uses λ- and μ-abstractions. After that, we extend Segerberg's system by **S5** and **S4** modalities: we consider necessity, possibility, impossibility (paracomplete negation), and unnecessity (paraconsistent negation). We use a modified version of Biermann and de Paiva's [16] rules: all our rules are general elimination and introduction rules.

Chapter 3 is devoted to proof theory for three- and four-valued logics. It is also a well-developed area of research; there are several methods of producing cut-free sequent calculi and natural deduction systems for these logics. We briefly discuss them in Chapter 3. We take one of the most general and fruitful methods, Kooi and Tamminga's correspondence analysis [96, 189, 97], originally developed for natural deduction. It uses the same ideas as Segerberg's [173] approach to classical logic and develops them further. It is a continuation of our previous research, mainly the paper [145], we generalise its results for a wider class of three-valued logics. At that, we prove the normalisation theorem for the considered natural deduction systems and show that they have negation subformula property, which is a rare case in the history of correspondence analysis (the normalisation theorem has been proved only for the formulation of intuitionistic logic with n-ary connectives obtained by Geuvers, van der Giessen, and Hurkens [61], the formulation of classical logic with n-ary connectives obtained by Geuvers and Hurkens [62], and a *labeled* natural deduction system for many-valued logics by Englander, Haeusler, and Pereira [38]). Then we transform natural deduction systems into sequent calculi and prove cut admissibility for them. As for four-valued logics, we provide a constructive proof of the cut admissibility theorem for Kooi and Tamminga's [97] sequent calculi for **FDE**-style four-valued logics (Kooi and Tamminga have only a semantic proof), then we extend their result and provide both types of proofs of cut admissibility for a wider class of **FDE**-style logics. Finally, we transform sequent calculi for four-valued logics into natural deduction systems for them and prove the normalisation theorem together with the negation subformula property.

Chapter 4 is devoted to modal three- and four-valued logics as well as to modal multilattice logics, which are an algebraic generalisation of modal four-valued **FDE**-style logics. There are several approaches to modal many-valued logics, but the most common one seems to be due to Fitting [52, 53], later further developed by Priest [164, 165], Odintsov and Wansing [137, 136], and other researchers (see Chapter 4 for more details). The novelty of our study is that we consider many-valued non-standard modalities and very generally formulated propositional many-valued parts of these logics: due to correspondence analysis, we need to choose some three- or four-valued negation, and then we are free to add any tabular three- or four-valued n-ary connective with the rules mechanically produced by correspondence analysis. Again, both sequent and natural deduction calculi are considered, cut admissibility, normalisation, and the negation subformula property are proven. Multilattice logic was introduced by Shramko [177] as a generalisation of Arieli and Avron's four-valued bilattice logic [2] (which in turn is a generalisation of Belnap [13, 14] and Dunn [34] **FDE** based on de Morgan lattices), Shramko and Wansing's sixteen-valued trilattice logic [179], Zaitsev's eight-valued tetralattice logic [198]. Modal multilattice logic was formulated by Kamide and Shramko [92] as an extension of multilattice logic by **S4**-style modalities. We consider a bit different extension of multilattice logic by **S4**-style modalities as well as its extension by **S5**-style modalities. The task of considering **S5**-style modalities was left as an open problem in [92]. We provide a cut-free hypersequent calculus for **S5**-style modal extension of multilattice logic (we consider both standard and non-standard modalities). Then we investigate the logics with weaker modalities (**K** and its reflexive, serial, symmetric, and transitive extensions): for them we use nested sequent calculi. For **S5**- and **S4**-style logics, we investigate natural deduction systems as well.

The results presented in Sections 2.1, 2.2, and 2.3 of Chapter 2 are mainly based on the results published in the author's paper [148]; the results of Sections 2.4 and 2.5 are previously unpublished. The results described in Sections 3.1, 3.2, 3.3, and 3.4 of Chapter 3 are mainly based on the joint paper of the author and Nils Kürbis [106] (to be more exact, Sections 3.1, 3.2 and 3.3 are entirely written by the author; in Section 3.4, Lemmas 98 and 99 are proven by N. Kürbis, the formulations of Definitions 95, 96, and 97 are due to N. Kürbis; as for the rest of Section 3.4, the author and N. Kürbis have contributed equally); the results of Sections 3.5 and 3.6 are previously unpublished. The results presented in Section 4.5 of Chapter 4 are partly published in the joint papers of the author and Oleg Grigoriev [71, 69] (to be more exact, the notions of Tarski, Kuratowski, and Halmos are taken from [69]; the formulation of the hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$ with standard modalities is the result from [71]. The author and Oleg Grigoriev have contributed equally to the papers [71, 69]); the results of Sections 4.1, 4.2, 4.3, and 4.4 are previously unpublished.

Chapter 2

Proof systems for selected modal logics

2.1 Preface

Modal logic is usually formulated in a language containing a necessity operator (denoted as $\Box$) and/or possibility operator (denoted as $\Diamond$).[1] However, in the literature one may find other modalities (and not just temporal, deontic, epistemic, etc.). One of them is the non-contingency operator (following Zolin [201], we denote it as $\rhd$) which can be defined as follows: $\rhd A = \Box A \vee \Box \neg A$. Thus, a proposition is non-contingent iff it is necessary or its negation is necessary. A contingency operator ($\blacktriangleright$ in our notation) is defined as follows: $\blacktriangleright A = \neg \rhd A = \Diamond A \wedge \Diamond \neg A$, i.e., a proposition is contingent iff it is possibly true and also possibly false.

Contingency may also be understood as 'ignorance' in epistemic logic [193] or a 'knowing whether' operator [40] which may be used to formalise some problems in AI [144] or microeconomics [73]. There are alternative interpretations of contingency: doxastic ('no belief' or 'undecided' [114]), deontic ('(moral) indifference' [197]), spatial ('topological border' [184]), and provability ('undecidable (in Peano Arithmetic)' [200]).

Although $\rhd$ and $\blacktriangleright$ are expressed in the standard modal language, since Montgomery and Routley [128, 129, 130] logicians have studied non-contingent and contingent versions of the well-known modal logics, i.e., the ones that contain $\rhd$ and $\blacktriangleright$ as primitive operators instead of $\Box$ and $\Diamond$. In many cases, their languages are less expressive than the standard one which makes the problem of their axiomatization non-trivial. Montgomery and Routley themselves formalised contingent and non-contingent logics based on **T**, **S4**, and **S5** via Hilbert-style calculi. The basic logics, contingent and non-contingent versions of **K**, have various axiomatizations developed by Humberstone [75], Kuhn [99], Zolin [203, 201], van der Hoek and Lomuscio [193]. Transitive and Euclidian contingent and non-contingent logics were formalised by Kuhn [99], Zolin [203], Steinsvold [184]. Fan [44, 45] paid special attention to symmetric logics. Probably the most impressive results were obtained in the case of reflexive non-contingent logics: Zolin [200] formulated a general method of constructing Hilbert-style calculi for them, using the fact that $\Box A = A \wedge \rhd A$.[2] However, the non-reflexive case is still non-trivial.[3] Surprisingly, from a proof-theoretic perspective, even the relatively simple reflexive case is problematic. Zolin [200, 201] developed sequent calculi for many non-contingent logics, including the **S5**-based one, but none of them is cut-free. This fact has inspired our attempt to present cut-free calculi for these logics with a more general framework of hypersequents rather than ordinary sequents. We would like to pay special attention to **S5**-style modal logics, since **S5** is known for

[1] The part of this Chapter devoted to **S5**-style logics is significantly based on the material from the author's paper [148].

[2] It is clear that his method may be adapted for contingent logics, since in reflexive logics it holds that $\Box A = A \wedge \neg \blacktriangleright A$ as well as $\Diamond A = \rhd A \to A$ and $\Diamond A = \neg \blacktriangleright A \to A$. In non-reflexive logics, these equivalences do not hold, but, as was shown by Pizzi [155], adding propositional quantifiers allow us to define necessity: $\Box A = \forall p(\rhd(p \wedge \rhd A) \to \rhd p)$.

[3] There are also studies of neighbourhood frames in the contingent language [42]. Among other variants of contingent logic, let us mention its combination with public announcement logic [32].

having plenty of cut-free hypersequent calculi. We choose Restall's [168] hypersequent calculus for **S5**, since it is one of the simplest calculi for this logic.[4] We will also consider logics weaker than **S5**, that is **K** and its reflexive, serial, transitive, and symmetric extensions. For these logics we will use even more general machinery than hypersequent calculi: nested sequent calculi in the spirit of Poggiolesi's results [157] for the logics with $\Box$. We do not consider Euclidean extensions of **K** because of some limitations of Poggiolesi's approach.

Aside from contingency and non-contingency, there are the concepts of essence and accident. A sentence is essentially true iff it is either false or necessarily true, i.e., if it is true, then it is necessarily true. A sentence is accidentally true iff it is true, but not necessarily true (i.e., its falsity is possible). Thus, the operators of essential and accidental truth (we denote them $\circ$ and $\bullet$, following Marcos [118]) are defined as follows: $\circ A = \neg A \vee \Box A = A \rightarrow \Box A$ and $\bullet A = \neg \circ A = A \wedge \neg \Box A = A \wedge \Diamond \neg A$. We should emphasize two points here. First, we follow Marcos' approach to essence and accidence, which is the de dicto one, while Fine [47, 48, 49] developed the de re approach. Second, as Gilbert and Venturi [63, p. 888] note, it is important not to conflate 'the notion of being accidentally/essentially true and the notion of being accidental/essential in the sense of being *mutable* or *immutable*'. They argue that Marcos deals with 'accidentally/essentially true', although he himself does not emphasise it. They introduce their own accident and essence operators: $\mathsf{A}A = \bullet A \vee \bullet \neg A$ and $\mathsf{E}A = \neg \mathsf{A}A = \circ A \vee \circ \neg A$.

> 'As for 'accidentally true' and 'essentially true', these can now be given straightforward formalizations as $A \wedge \mathsf{A}A$ and $A \wedge \mathsf{E}A$, respectively.' <...> 'As a final remark, one might wonder what the logic of these new operators is. But the logic of A and E is the logic of $\circ$ and $\bullet$, because all four of these operators are interdefinable (one can define $\bullet A$ as $A \wedge \mathsf{A}A$, as we mentioned above). Therefore, our ultimate claim is that the formal framework for exploring notions of essence and accident proposed by Marcos in [118] is a good one, but more precision is required to separate, and formalize, all of the desirable concepts within this sphere.' [63, p. 890, the notation adjusted]

We consider the modalities $\circ$ and $\bullet$, since the logic of A and E is reducible to them. Additionally, we consider 'accidentally/essentially *false*' modalities, denoting them as $\widetilde{\circ}$ and $\widetilde{\bullet}$, respectively, and defining them as $\widetilde{\circ}A = \circ\neg A = \neg A \rightarrow \Box \neg A$ and $\widetilde{\bullet}A = \bullet \neg A = \neg A \wedge \Diamond A$. So a proposition is essentially false iff its falsity implies the necessity of its falsity, and a proposition is accidentally false iff it is false, but its truth is possible.

In accidentally/essentially true logics, we have the following equalities in the case of serial frames: $\Box A = A \wedge \neg \bullet A$, $\Diamond A = \neg \bullet A \rightarrow A$, $\Box A = A \wedge \circ A$, and $\Diamond A = \circ A \rightarrow A$ [41]. It simplifies the task of providing an axiomatization of these logics, but the non-serial case is non-trivial as well as well-behaved (hyper)sequent calculi have not been developed for these logics (to the best of our knowledge). As for Hilbert-style calculi for them, see [41, 183, 118, 43]. In [43, 46] a combination of accident and contingent logics is suggested. Labelled (i.e., using explicit semantic elements) analytic tableaux were developed by Venturi and Yago [192] for essence and contingent logics. Notice that our calculi do not have any explicit semantic elements (see Poggiolesi [157] for the advantages of such calculi). Let us also mention that the very notion of accidental truth was used by Small [181] in the context of Gödel's ontological proof.

The idea to formulate a paraconsistent logic over **S5** is due to Jaśkowski [85]. Taking his inspiration from Jaśkowski's work, Béziau [15] presented a paraconsistent logic **Z** which is the result of the replacement of Boolean negation in classical logic with a paraconsistent one defined as negated necessity. Thus, we have ${\sim}A = \neg \Box A$, where $\sim$ is paraconsistent negation, as well as $\Box A = \neg{\sim}A$ and

[4] Pioneers in the development of hypersequent calculi in general and for **S5** in particular are Mints [126], Pottinger [160], and Avron [4]. Later on, different hypersequent calculi for **S5** were presented by Poggiolesi [156], Lahav [110], Kurokawa [107], Restall [168], Bednarska and Indrzejczak [12], and Indrzejczak [83]. See [12, 84] for a survey and comparison of these calculi. See also recent Mohammadi and Aghaei's paper [127] on rooted hypersequent calculi for **S5**.

$\Diamond A = \sim\neg A$. Marcos [118] generalized Béziau's approach: he considered paraconsistent logics based on modal logics which are weaker than **S5** and investigated a paracomplete negation (we denote it $\dot{\sim}$) defined as $\dot{\sim}A = \neg\Diamond A$ (hence, $\Box A = \dot{\sim}\neg A$ and $\Diamond A = \neg\dot{\sim}A$). Avron and Lahav [9] developed a cut-free hypersequent calculus for **Z** which is similar to Restall's one for **S5**. We mention their calculus in the next sections (and additionally present a constructive cut elimination proof for it) to make our study more complete. Moreover, we introduce a related calculus for a paracomplete version of **Z** which we call $\dot{\mathbf{Z}}$ and is understood as the result of the replacement of Boolean negation in classical logic with the paracomplete negation $\dot{\sim}$. Coniglio and Prieto-Sanabria [29] formulated a paraconsistent logic **LTop** with $\sim$ (at that Boolean negation was left in the language) on the basis of **S4**. We formulate a cut-free nested sequent calculus for it. For a systematic study of proof systems for logics with negative modalities, see the paper [109] by Lahav, Marcos and Zohar. However, it is not the case that all the calculi presented there are cut-free.

As for natural deduction systems for modal logics, to the best of our knowledge, the existing research is devoted (mainly) to consideration of logics with $\Box$ and $\Diamond$. At that point, there are several different approaches to modal natural deduction. In general, in the case of natural deduction systems we can distinguish two main approaches: Jaśkowski-Fitch-style linear format and Gentzen-style tree-format; one may also consider Suppes-style linear format as a separate approach. In the case of modal natural deduction, an additional classification can be done: following Indrzejczak [77], we can divide natural deduction systems based on the modalization of assumptions, the modalization of rules, the modalization of the reiteration rule, and the application of modal assumptions, which is a variant of modalization of the reiteration rule. Among the representatives of the first approach (modalization of assumptions) are Curry [31], Borkowski and Słupecki [20], Prawitz [161], Corcoran [30], Satre [171]. At that Curry and Prawitz followed Gentzen-style tree format, while Borkowski, Słupecki, and Corcoran applied Jaśkowski-Fitch-style linear format, Satre used Suppes-style format. The approach based on modalization of rules is due to Bull and Segerberg [23]; they use Jaśkowski-Fitch-style linear format. The approach based on modalization of the reiteration rule is due to Fitch [50, 51], it is based on Jaśkowski-Fitch-style linear format, and seems to be the most popular way of constructing natural deduction for modal logics. Fitch's approach can be applied to many modal logics; it was further developed by Siemens [176] and generalised by Fitting [54] whose approach is the application of modal assumptions. One may find more details in [77].

We just observed that an advantage of tree format is the possibility of considering the normalisation theorem, which is especially important for us, and that is why we choose such an approach. The first attempt to prove normalisation for modal logics is due to Prawitz [161]: he considered **S4** and **S5** as well as their first-order and minimal and intuitionistic versions. However, later on, some mistakes were found in his proofs by Medeiros and Da Paz [120]. Among the attempts to provide the proof of normalisation for modal logics, we would like to emphasise Biermann and de Paiva's [16] approach, which gives a simple formulation of the rules for intuitionistic **S4** and proves normalisation for it. Later on, Kürbis [100] considers their rules in the framework of classical **S4** and shows how to adapt them for **S5**. A disadvantage of tree format in the case of modal logics and, more generally, of the approach based on modalization of assumptions is its limited scope of application: it works only for **S4** and **S5**. As a result, we will consider natural deduction systems only for these two logics.

2.2 Preliminaries: semantics, Hilbert-style systems, and ordinary sequent calculi

Let $\clubsuit \in \{\Box, \Diamond, \rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$, $\mathcal{P}$ be a set $\{p_0, p_1, \ldots\}$ of propositional variables, $\mathscr{A}$ be the alphabet $\langle \mathcal{P}, \clubsuit, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, (,) \rangle$. We fix a modal language $\mathscr{L}_\clubsuit$ with the alphabet $\mathscr{A}$ which forms the set $\mathscr{F}_\clubsuit$ of all $\mathscr{L}_\clubsuit$-formulas in a standard inductive way. In some cases we use bimodal languages, e.g. $\mathscr{L}_{\Box\Diamond}$, defined in an analogous way.

Consider the language $\mathscr{L}_{\Box\Diamond}$. Let us describe semantics of the modal logic **K** and its extensions. A triple $\langle W, R, \vartheta\rangle$ is said to be an **K**-model iff $W \neq \emptyset$, $R \subseteq W \times W$, and ϑ is a mapping from $W \times \mathscr{F}_{\Box\Diamond}$ to $\{1, 0\}$ such that it preserves classical conditions for truth-value connectives and for any $A \in \mathscr{F}_{\Box\Diamond}$ and $x \in W$ we have:

– $\vartheta(\Box A, x) = 1$ iff $\forall_{y\in W}(R(x, y)$ implies $\vartheta(A, y) = 1)$,

– $\vartheta(\Diamond A, x) = 1$ iff $\exists_{y\in W}(R(x, y)$ and $\vartheta(A, y) = 1)$.

A formula A is true in a world $w \in W$ iff $\vartheta(A, w) = 1$. A formula A follows from the set of formulas Γ ($\Gamma \models_{\mathbf{K}} A$) iff for every **K**-model $\langle W, R, \vartheta\rangle$ and every $w \in W$, if any $B \in \Gamma$ is true in w, then A is true in w. A formula is **K**-valid iff it follows from the empty set of formulas.

A Hilbert-style calculus for **K** contains the subsequent axioms (to be more exact, schemes of axioms) and rules:

– all schemes of axioms of classical propositional logic,

– $\Box(A \to B) \to (\Box A \to \Box B)$ (the **K**-axiom),

– $\Diamond A \leftrightarrow \neg\Box\neg A$,

– $\dfrac{A \quad A \to B}{B}$ (modus ponens),

– $\dfrac{\vdash_{\mathbf{K}} A}{\vdash_{\mathbf{K}} \Box A}$ (Gödel's rule or necessitation rule).

The notion of the proof is defined in a standard way as a sequence of formulas that are either axioms or are obtained from the previous ones by the rules.

Before we introduce a sequent calculus for **K**, let us declare that following Restall [168] by a sequent, we understand a pair written as $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas[5] Let us recall the formulation of a sequent calculus for classical propositional logic **CPL**[6]. The calculus has the following axiom: $A \Rightarrow A$. Its structural rules are as follows (contraction, weakening, and cut):

$$(\mathrm{C}{\Rightarrow})\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad ({\Rightarrow}\mathrm{C})\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} \qquad (\mathrm{W}{\Rightarrow})\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad ({\Rightarrow}\mathrm{W})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

$$(\mathrm{Cut})\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Theta \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda}$$

The logical rules are as follows:

$$(\neg \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg)\ \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$$

$$(\wedge \Rightarrow)\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$(\vee \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

[5] Alternatively, one could use sets of formulas or lists of formulas, which would lead to the other choice of structural rules: the option with sets makes contraction rules superfluous, and the option with lists of formulas requires exchange rules.

[6] There are several options; we use the restriction for the case of sequents of Restall's hypersequent calculus [168], which plays an important role in the further consideration connected with the study of **S5**-style logics.

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Theta \Rightarrow \Lambda}{A \rightarrow B, \Gamma, \Theta \Rightarrow \Delta, \Lambda} \qquad (\Rightarrow\rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

$$(\leftrightarrow\Rightarrow)\ \frac{B, \Gamma \Rightarrow \Delta, A \quad A, \Theta \Rightarrow \Lambda, B}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, A \leftrightarrow B} \qquad (\Rightarrow\leftrightarrow)\ \frac{A, B, \Gamma \Rightarrow \Delta \quad \Theta \Rightarrow \Lambda, A, B}{A \leftrightarrow B, \Gamma, \Theta \Rightarrow \Delta, \Lambda}$$

Definition 1 (Proof)**.** By a proof in the above presented sequent calculus for **CPL** we mean a tree which nodes are sequents such that leaves are axioms and other nodes are obtained from the upper ones by applications of the rules of the calculus. We write $\vdash_{\mathbf{CPL}} S$ iff there is a proof of a sequent S in the sequent calculus for **CPL**.

Definition 2 (An admissible and a derivable rule)**.** A rule $\frac{S_1,\ldots,S_n}{S}$, where $S_1, \ldots S_n, S$ are sequents is *admissible* in a sequent calculus **SC** iff $\vdash_{\mathbf{SC}} S_1, \ldots, \vdash_{\mathbf{SC}} S_n$ implies $\vdash_{\mathbf{SC}} S$, and *derivable* in a sequent calculus **SC** iff $S_1, \ldots, S_n \vdash_{\mathbf{SC}} S$.

The rule (Cut) is admissible in the above-presented calculus.

As for a sequent calculus for **K**, it can be obtained from the one for classical propositional logic by adding the rule $(\Rightarrow_{\mathbf{K}} \Box)$, if **K** is formulated in the language $\mathscr{L}_{\Box}$; the rule $(\Diamond \Rightarrow_{\mathbf{K}})$, if **K** is formulated in the language $\mathscr{L}_{\Diamond}$; and the rules $(\Rightarrow^*_{\mathbf{K}} \Box)$ and $(\Diamond \Rightarrow_{\mathbf{K}^*})$, if **K** is formulated in the language $\mathscr{L}_{\Box\Diamond}$:

$$(\Rightarrow_{\mathbf{K}} \Box)\ \frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} \qquad (\Diamond \Rightarrow_{\mathbf{K}})\ \frac{A \Rightarrow \Delta}{\Diamond A \Rightarrow \Diamond\Delta}$$

$$(\Rightarrow^*_{\mathbf{K}} \Box)\ \frac{\Gamma \Rightarrow \Delta, A}{\Box\Gamma \Rightarrow \Diamond\Delta, \Box A} \qquad (\Diamond \Rightarrow^*_{\mathbf{K}})\ \frac{A, \Gamma \Rightarrow \Delta}{\Diamond A, \Box\Gamma \Rightarrow \Diamond\Delta}$$

If we formulate **K** in the language $\mathscr{L}_{\Box\Diamond}$ and instead of $(\Rightarrow^*_{\mathbf{K}} \Box)$ and $(\Diamond \Rightarrow_{\mathbf{K}^*})$ will use $(\Rightarrow_{\mathbf{K}} \Box)$ and $(\Diamond \Rightarrow_{\mathbf{K}})$, then we fail to prove the formulas $\Box A \leftrightarrow \neg\Diamond\neg A$ and $\Diamond A \leftrightarrow \neg\Box\neg A$. This peculiarity was noticed by Kripke [98].

The extensions of the modal logic **K** are obtained by the restriction of the accessibility relation R (from the semantic point of view) or adding to the Hilbert-style calculus for **K** axioms (or schemes of axioms) corresponding to these semantic restrictions (from the syntactical point of view). The most popular conditions for R are as follows (we give them together with the axioms corresponding to them):

- reflexivity, $\forall x R(x, x)$; **T**-axiom, $\Box A \rightarrow A$;
- seriality, $\forall x \exists y R(x, y)$; **D**-axiom, $\Box A \rightarrow \Diamond A$;
- symmetry, $\forall x, y (R(x, y)$ implies $R(y, x))$; **B**-axiom, $A \rightarrow \Box\Diamond A$;
- transitivity, $\forall x, y, z\big((R(x, y)$ and $R(y, z))$ implies $R(x, z)\big)$; **4**-axiom, $\Box A \rightarrow \Box\Box A$;
- Euclideaness, $\forall x, y, z\big((R(x, y)$ and $R(x, z))$ implies $R(y, z)\big)$; **5**-axiom, $\Diamond A \rightarrow \Box\Diamond A$.

The combinations of these conditions (axioms) and the logic **K** itself can produce only 15 different logics: **K**, **D**, **T**, **K4**, **D4**, **S4** (=**KT4**), **K5**, **D5**, **K45**, **D45**, **KB**, **DB**, **TB**, **KB45**, and **S5** (=**KT5**) (see, e.g. [28]). From the names of these logics it is possible to extract which axioms should be added to **K** to obtain them. Notice that in the case of **S5** all these five conditions hold and they can be replaced with the universality condition: $\forall x, y R(x, y)$. In the other words, $R = W \times W$. As a consequence, in **S5** we may consider models which are pairs of the form $\langle W, \vartheta\rangle$ and truth conditions for $\Box$ and $\Diamond$, being transformed as follows:

- $\vartheta(\Box A, x) = 1$ iff $\forall_{y\in W}\vartheta(A, y) = 1$,
- $\vartheta(\Diamond A, x) = 1$ iff $\exists_{y\in W}\vartheta(A, y) = 1$.

To obtain a sequent calculus for **T** we need to add the following rules to $\mathbf{K}$[7]:

$$(\Box \Rightarrow_{\mathbf{T}})\ \frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow_{\mathbf{T}} \Diamond)\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \Diamond A}$$

In the description of sequent calculi for other logics, we follow Fitting's approach [54] in Indrzejczak's presentation [77]. Consider the following rules:

$$(\mathbf{D} \Rightarrow)\ \frac{\Gamma^\star \Rightarrow \Delta^\natural}{\Gamma \Rightarrow \Delta} \qquad (\Rightarrow \Box^\star_\natural)\ \frac{\Gamma^\star \Rightarrow \Delta^\natural, A}{\Gamma \Rightarrow \Delta, \Box A} \qquad (\Diamond \Rightarrow^\star_\natural)\ \frac{A, \Gamma^\star \Rightarrow \Delta^\natural}{\Diamond A, \Gamma \Rightarrow \Delta}$$

where $\Gamma^\star$ and $\Delta^\natural$ are defined as follows, depending on the logic which is under consideration:

Logic	$\Gamma^\star$	$\Delta^\natural$
K, **T**, **D**	$\{C \mid \Box C \in \Gamma\}$	$\{D \mid \Diamond D \in \Delta\}$
S4	$\{\Box C \mid \Box C \in \Gamma\}$	$\{\Diamond D \mid \Diamond D \in \Delta\}$
K4, **D4**	$\{C \mid \Box C \in \Gamma\} \cup \{\Box C \mid \Box C \in \Gamma\}$	$\{D \mid \Diamond D \in \Delta\} \cup \{\Diamond D \mid \Diamond D \in \Delta\}$
KB, **TB**, **DB**	$\{C \mid \Box C \in \Gamma\} \cup \{\Diamond C \mid C \in \Gamma\}$	$\{D \mid \Diamond D \in \Delta\} \cup \{\Box D \mid D \in \Delta\}$
S5	$\{\Box C \mid \Box C \in \Gamma\} \cup \{\Diamond C \mid \Diamond C \in \Gamma\}$	$\{\Diamond D \mid \Diamond D \in \Delta\} \cup \{\Box D \mid \Box D \in \Delta\}$
KB4	$\{C \mid \Box C \in \Gamma\} \cup \{\Box C \mid \Box C \in \Gamma\} \cup \{\Diamond C \mid C \in \Gamma\}$	$\{D \mid \Diamond D \in \Delta\} \cup \{\Diamond D \mid \Diamond D \in \Delta\} \cup \{\Box D \mid D \in \Delta\}$
K5, **D5**	$\{C \mid \Box C \in \Gamma\} \cup \{\Diamond C \mid \Diamond C \in \Gamma\}$	$\{D \mid \Diamond D \in \Delta\} \cup \{\Box D \mid \Box D \in \Delta\}$
K45, **D45**	$\{C \mid \Box C \in \Gamma\} \cup \{\Box C \mid \Box C \in \Gamma\} \cup \{\Diamond C \mid C \in \Gamma\} \cup \{\Diamond C \mid \Diamond C \in \Gamma\}$	$\{C \mid \Box C \in \Gamma\} \cup \{\Box C \mid \Box C \in \Gamma\} \cup \{C \mid \Diamond C \in \Gamma\} \cup \{\Diamond C \mid \Diamond C \in \Gamma\}$

Each of the 15 logics has the rules $(\Rightarrow \Box^\star_\natural)$ and $(\Diamond \Rightarrow^\star_\natural)$; if the logic in question is serial, it contains the rule $(\mathbf{D} \Rightarrow)$ as well; if the logic in question is reflexive, it contains the rules $(\Rightarrow \Box_{\mathbf{T}})$ and $(\Diamond \Rightarrow_{\mathbf{T}})$ as well.

As Takano pointed out [188], these modal logics could be divided into three groups depending on the properties of their sequent calculi. The first group consists of **K**, **D**, **T**, **K4**, **D4**, **S4**, **K45**, and **D45**. These logics "have sequent calculi with the cut-elimination property (and so the subformula property)" [188, p. 116]. The second group contains the logics **KB**, **DB**, **TB**, **KB4**, and **S5** which "have sequent calculi with the subformula property but without the cut-elimination property" [188, p. 116]. The third group consists of **K5** and **D5**. They have sequent calculi without cut elimination and subformula property. However, Takano presents a modified subformula property for these logics.

As for natural deduction systems for modal logics, let us present the systems for **S4** and **S5** only, since only these logics have tree-format natural deduction systems. First of all, let us introduce some natural deduction system for classical propositional logic. We are going to describe two such systems: one introduced by Gentzen [59] and another by Milne [125]. The former calculus is an example of a natural deduction system with ordinary introduction and elimination rules (except disjunction elimination, which is a general elimination rule), while the latter calculus is an example of a natural deduction system with general introduction and general elimination rules only. Note also that Milne's system has been used by Kürbis [103] for his normalisation proof.

Gentzen's system *NK* (to be more exact, its propositional fragment) contains the following inference rules, where $\bot$ stands for a constant falsum:

$$(\vee I_1)\ \frac{\begin{matrix}\mathfrak{D}\\ A\end{matrix}}{A \vee B} \qquad (\vee I_2)\ \frac{\begin{matrix}\mathfrak{D}\\ B\end{matrix}}{A \vee B} \qquad (\vee E^{a,b})\ \frac{\begin{matrix}\\ \mathfrak{D}_0\\ A \vee B\end{matrix} \quad \begin{matrix}[A]^a\\ \mathfrak{D}_1\\ C\end{matrix} \quad \begin{matrix}[B]^b\\ \mathfrak{D}_2\\ C\end{matrix}}{C}$$

[7]If **K** is formulated in $\mathscr{L}_\Box$, we add $(\Box \Rightarrow_{\mathbf{T}})$; if in $\mathscr{L}_\Diamond$, we add $(\Rightarrow_{\mathbf{T}} \Diamond)$; if in $\mathscr{L}_{\Box\Diamond}$, we add both rules, due to the rules $(\Rightarrow^*_{\mathbf{K}} \Box)$ and $(\Diamond \Rightarrow_{\mathbf{K}^*})$ we do not need any special version of $(\Box \Rightarrow_{\mathbf{T}})$ and $(\Rightarrow_{\mathbf{T}} \Diamond)$.

$$(\wedge I)\ \frac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ A & B\end{matrix}}{A \wedge B} \qquad (\wedge E_1)\ \frac{\begin{matrix}\mathfrak{D}\\ A \wedge B\end{matrix}}{A} \qquad (\wedge E_2)\ \frac{\begin{matrix}\mathfrak{D}\\ A \wedge B\end{matrix}}{B} \qquad (\to I)^a\ \frac{\begin{matrix}[A]^a\\ \mathfrak{D}\\ B\end{matrix}}{A \to B} \qquad (\to E)\ \frac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ A \to B & A\end{matrix}}{B}$$

$$(\bot I)\ \frac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ A & \neg A\end{matrix}}{\bot} \qquad (\bot E)\ \frac{\begin{matrix}\mathfrak{D}\\ \bot\end{matrix}}{A} \qquad (\neg I)^a\ \frac{\begin{matrix}[A]^a\\ \mathfrak{D}\\ \bot\end{matrix}}{\neg A} \qquad (\neg E)\ \frac{\begin{matrix}\mathfrak{D}\\ \neg\neg A\end{matrix}}{A}$$

As Gentzen [59] notes, the rule $(\neg E)$ can be replaced with an axiom $A \vee \neg A$. If one deletes $(\neg E)$, then one gets a natural deduction system for intuitionistic logic.

Definition 3 (Deduction in *NK*)**.**

1. The formula occurrence A is a deduction in *NK* of A from the undischarged assumption A.
2. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are deductions in *NK*, then the applications of the above-mentioned rules are deductions of B in *NK* from the undischarged assumptions in $\mathfrak{D}_1$ and $\mathfrak{D}_2$ apart from those in the assumption classes a and b, which are discharged.
3. Nothing else is a deduction in *NK*.

Milne's natural deduction system has the following inference rules:

$$(\wedge I)^a\ \frac{\begin{matrix} & & [A \wedge B]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3\\ A & B & C\end{matrix}}{C} \qquad (\wedge E)^{a,b}\ \frac{\begin{matrix} & [A]^a[B]^b\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ A \wedge B & C\end{matrix}}{C}$$

$$(\vee I_1)^a\ \frac{\begin{matrix} & [A \vee B]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ A & C\end{matrix}}{C} \qquad (\vee I_2)^a\ \frac{\begin{matrix} & [A \vee B]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ B & C\end{matrix}}{C} \qquad (\vee E)^{a,b}\ \frac{\begin{matrix} & [A]^a & [B]^b\\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3\\ A \vee B & C & C\end{matrix}}{C}$$

$$(\to I)^a\ \frac{\begin{matrix} & [A \to B]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ B & C\end{matrix}}{C} \qquad (\mathrm{TR})^{a,b}\ \frac{\begin{matrix}[A]^a & [A \to B]^b\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ C & C\end{matrix}}{C} \qquad (\to E)^a\ \frac{\begin{matrix} & & [B]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3\\ A \to B & A & C\end{matrix}}{C}$$

$$(\neg I)^{a,b}\ \frac{\begin{matrix}[A]^a & [\neg A]^b\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ C & C\end{matrix}}{C} \qquad (\neg E)\ \frac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ \neg A & A\end{matrix}}{C}$$

The notion of a deduction is defined in a similar way to how it is defined in Gentzen's *NK*. As we can see, general introduction rules for a connective $*$ "instead of introducing formulas with a connective $*$ as main operator as the conclusion of the rule, they permit discharge of assumptions of that form" [104, p. 14226] and as for their conclusions they have formulas derivable from the formulas with a connective $*$ as main operator. Similarly, general elimination rules for a connective $*$ instead of having as their conclusions some subformulas of formulas with a connective $*$ as main operator have some formulas derivable from subformulas of formulas with a connective $*$.

As for natural deduction modal rules, we use the rules suggested by Bierman and de Paiva [16] as a modification of Prawitz's [161] ones more suitable for proving the normalisation theorem.

$$
(\Box I)^a \ \dfrac{\begin{matrix}\mathfrak{D}_1 \ldots \mathfrak{D}_m \\ B_1 \ldots B_m\end{matrix} \qquad \begin{matrix}[B_1 \ldots B_m]^a \\ \mathfrak{D} \\ A\end{matrix}}{\Box A} \qquad (\Box E)\ \dfrac{\begin{matrix}\mathfrak{D} \\ \Box A\end{matrix}}{A}
$$

$$
(\Diamond I)\ \dfrac{\begin{matrix}\mathfrak{D} \\ A\end{matrix}}{\Diamond A} \qquad (\Diamond E)^a\ \dfrac{\begin{matrix}\mathfrak{D}_0 \\ \Diamond A\end{matrix} \quad \begin{matrix}\mathfrak{D}_1 \ldots \mathfrak{D}_m \\ B_1 \ldots B_m\end{matrix} \qquad \begin{matrix}[A, B_1 \ldots B_m]^a \\ \mathfrak{E} \\ C\end{matrix}}{C}
$$

$B_1, \ldots, B_m$ are exactly the undischarged assumptions in $\mathfrak{D}$ in $(\Box I)$ and $A, B_1, \ldots, B_m$ in $\mathfrak{E}$ in $(\Diamond E)$. For **S4**, $B_1, \ldots, B_m$ are required to be of the form $\Box D_1, \ldots, \Box D_m$ and C to be of the form $\Diamond D$. For **S5**, $A, B_1, \ldots, B_m, C$ are required to be modalized. An adaptation of these rules for **S5** was given by Kürbis [100].

Biermann and de Paiva [16] originally formulated their rules for intuitionistic **S4** with both $\Box$ and $\Diamond$ in the language. In classical **S4** built in this language, it seems impossible to prove the formulas $\Box A \to \neg\Diamond\neg A$, $\Diamond A \to \neg\Box\neg A$, $\neg\Diamond\neg A \to \Box A$, and $\neg\Box\neg A \to \Diamond A$ with these rules (intuitionistic **S4** does not have these formulas as its provable formulas in contrast to classical **S4**). So we consider classical **S4** built in the language with $\Box$, but without $\Diamond$. Biermann and de Paiva's rules for $\Box$ are sound and complete in the classical case as well. As for the rules for $\Diamond$, since they involve the use of $\Box$ in the language and we cannot have both $\Box$ and $\Diamond$ in classical **S4**, we do not consider the rules for $\Diamond$ in **S4**. In **S5**, the proviso for the rules is more liberal ($A, B_1, \ldots, B_m, C$ are required to be modalized), so we can consider the rules for $\Diamond$ in classical **S5**. However, again, we cannot have both modalities in the language.

We write $\mathbf{L}^{\triangleright}$ for the non-contingency version of the logic $\mathbf{L}$, i.e., the logic over $\mathbf{L}$-models in the language $\mathscr{L}_{\triangleright}$. Thus, we have the case for $\triangleright$ instead of the cases for $\Box$ and $\Diamond$:

– $V(\triangleright A, x) = 1$ iff $\forall_{y \in W}(R(x, y)$ implies $V(A, y) = 1)$ or $\forall_{y \in W}(R(x, y)$ implies $V(A, y) = 0)$.

In the case of **S5**, this condition can be simplified:

– $V(\triangleright A, x) = 1$ iff $\forall_{y \in W} V(A, y) = 1$ or $\forall_{y \in W} V(A, y) = 0$.

The Hilbert-style calculi for reflexive non-contingency logics were developed by Montgomery and Routley [128, 129, 130] and Zolin [200]. The Hilbert-style calculus for the logic $\mathbf{T}^{\triangleright}$ contains the following axioms and rules (the names of the axioms are due to Zolin [201, 203]):

– all schemes of axioms of classical propositional logic,

– $A \to \big(\triangleright(A \to B) \to (\triangleright A \to \triangleright B)\big)$ (weak distributivity),

– $\triangleright A \leftrightarrow \triangleright\neg A$ (mirror axiom),

– $\dfrac{A \quad A \to B}{B}$ (modus ponens),

– $\dfrac{\vdash A}{\vdash \triangleright A}$ (Gödel's rule for $\triangleright$).

The extensions of $\mathbf{T}^{\triangleright}$ can be formalised as follows:

– $\mathbf{TB}^{\triangleright} = \mathbf{T}^{\triangleright} + A \to \triangleright(\triangleright A \to A)$,

– $\mathbf{S4}^{\triangleright} = \mathbf{T}^{\triangleright} + \triangleright A \to \triangleright\triangleright A$,

– $\mathbf{S5}^{\triangleright} = \mathbf{T}^{\triangleright} + \triangleright\triangleright A$.

$\mathbf{K}^{\rhd}$ was first axiomatized by Humberstone [75], $\mathbf{K4}^{\rhd}$ by Kuhn [99]. We give Zolin's axiomatization [203, 201] for them and $\mathbf{K5}^{\rhd}$. The Hilbert-style calculus for the logic $\mathbf{K}^{\rhd}$ contains the following axioms and rules:

- all schemes of axioms of classical propositional logic,
- $\rhd(A \leftrightarrow B) \to (\rhd A \leftrightarrow \rhd B)$ (equivalence),
- $\rhd A \leftrightarrow \rhd\neg A$ (mirror axiom),
- $\rhd A \to \big(\rhd(B \to A) \vee (A \to C)\big)$ (dichotomy),
- $\dfrac{A \quad A \to B}{B}$ (modus ponens),
- $\dfrac{\vdash A}{\vdash \rhd A}$ (Gödel's rule for $\rhd$).

The extensions of $\mathbf{K}^{\rhd}$ can be formalised as follows:

- $\mathbf{K4}^{\rhd} = \mathbf{K}^{\rhd} + \rhd A \to \rhd(B \to \rhd A)$ (weak transitivity),
- $\mathbf{K5}^{\rhd} = \mathbf{K}^{\rhd} + \neg\rhd A \to \rhd(B \to \neg\rhd A)$ (weak Euclideanness),
- $\mathbf{K45}^{\rhd} = \mathbf{K4}^{\rhd} + \neg\rhd A \to \rhd(B \to \neg\rhd A)$.

The Hilbert-style calculus for $\mathbf{KB}^{\rhd}$ was developed by Fan, Wang, and Ditmarsch [39]:

- all schemes of axioms of classical propositional logic,
- $\big(\rhd(C \to A) \wedge \rhd(\neg C \to A)\big) \to \rhd A$,
- $\rhd A \to \big(\rhd(A \to B) \vee \rhd(\neg A \to C)\big)$,
- $\rhd A \leftrightarrow \rhd\neg A$,
- $A \to \rhd\Big(\big(\rhd A \wedge \rhd(A \to B) \wedge \neg\rhd B\big) \to C\Big)$,
- $\dfrac{A \quad A \to B}{B}$,
- $\dfrac{\vdash A \leftrightarrow B}{\vdash \rhd A \leftrightarrow \rhd B}$.

Gödel's rule for $\rhd$ is admissible in this system.

As follows from [203], serial non-contingency logics coincide with their non-serial companions, so $\mathbf{L}^{\rhd} = \mathbf{LD}^{\rhd}$, for any logic $\mathbf{L}$.

Non-cut-free sequent calculi for non-contingency logics were developed by Zolin [200, 201]. To obtain sequent calculi for reflexive non-contingency logics, one needs to extend a sequent calculus for classical propositional logic by the following rules [201]:

$$(\Rightarrow^{\rhd 0}_{\mathbf{T}})\ \frac{A, \Gamma \Rightarrow \Delta}{\Gamma, \rhd\Gamma, \rhd\Delta \Rightarrow \Delta, \rhd A} \qquad (\Rightarrow^{\rhd 1}_{\mathbf{T}})\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma, \rhd\Gamma, \rhd\Delta \Rightarrow \Delta, \rhd A}$$

$$(\Rightarrow^{\rhd 0}_{\mathbf{S4}})\ \frac{A, \Gamma, \rhd\Gamma, \rhd\Delta \Rightarrow \Delta}{\Gamma, \rhd\Gamma, \rhd\Delta \Rightarrow \Delta, \rhd A} \qquad (\Rightarrow^{\rhd 1}_{\mathbf{S4}})\ \frac{\Gamma, \rhd\Gamma, \rhd\Delta \Rightarrow \Delta, A}{\Gamma, \rhd\Gamma, \rhd\Delta \Rightarrow \Delta, \rhd A}$$

$$(\Rightarrow^{\rhd 0}_{\mathbf{S5}})\ \frac{A, \Gamma, \rhd\Gamma, \rhd\Delta \Rightarrow \Delta, \rhd\Lambda}{\Gamma, \rhd\Gamma, \rhd\Delta \Rightarrow \Delta, \rhd\Lambda, \rhd A} \qquad (\Rightarrow^{\rhd 1}_{\mathbf{S5}})\ \frac{\Gamma, \rhd\Gamma, \rhd\Delta \Rightarrow \Delta, \rhd\Lambda, A}{\Gamma, \rhd\Gamma, \rhd\Delta \Rightarrow \Delta, \rhd\Lambda, \rhd A}$$

$$(\Rightarrow^{\rhd 0}_{\mathbf{TB}})\ \frac{\{A, \Gamma, \Phi' \Rightarrow \Phi, \rhd\Psi', \rhd\Psi, \Delta\}^{\Sigma=\Phi\cup\Psi}_{\Sigma'=\Phi'\cup\Psi'}}{\Gamma, \rhd\Gamma, \rhd\Delta, \Sigma' \Rightarrow \Delta, \Sigma, \rhd A} \qquad (\Rightarrow^{\rhd 1}_{\mathbf{TB}})\ \frac{\{\Gamma, \Phi' \Rightarrow \Phi, \rhd\Psi', \rhd\Psi, \Delta, A\}^{\Sigma=\Phi\cup\Psi}_{\Sigma'=\Phi'\cup\Psi'}}{\Gamma, \rhd\Gamma, \rhd\Delta, \Sigma' \Rightarrow \Delta, \Sigma, \rhd A}$$

As for non-reflexive logics, Zolin offered the following non-cut-free sequent calculi for $\mathbf{K}^{\rhd}$ and $\mathbf{K4}^{\rhd}$ (obviously, they contain rules for classical propositional logic).

$$(\Rightarrow^{\rhd}_{\vee})\ \frac{\Gamma \Rightarrow \Delta, \rhd A}{\Gamma \Rightarrow \Delta, \rhd(B \to A), \rhd(A \to C)} \qquad (\Rightarrow^{\rhd}_{\leftrightarrow})\ \frac{\Gamma, A \Rightarrow B, \Delta \quad \Gamma, B \Rightarrow A, \Delta}{\Gamma, \rhd A \Rightarrow \rhd B, \Delta}$$

$$(\Rightarrow^{\rhd}_{\mathbf{K}})\ \frac{\Gamma \Rightarrow A}{\rhd(\Gamma \vee A) \Rightarrow \rhd A} \qquad (\Rightarrow^{\rhd}_{\mathbf{K4}})\ \frac{\Gamma, \rhd\Lambda \Rightarrow A}{\rhd(\Gamma \vee A), \rhd\Lambda \Rightarrow \rhd A}$$

Zolin also developed sequent calculi for $\mathbf{GL}^{\rhd}$ and $\mathbf{Grz}^{\rhd}$ [200, 201]. He [202] formulated as a hypothesis sequent rules for $\mathbf{K5}^{\rhd}$ and $\mathbf{K45}^{\rhd}$, but did not establish, if they were enough for completeness.

$$(\Rightarrow^{\rhd}_{\mathbf{K5}})\ \frac{\Gamma \Rightarrow \rhd\Delta, A}{\rhd(\Gamma \vee A) \Rightarrow \rhd\Delta, \rhd A} \qquad (\Rightarrow^{\rhd}_{\mathbf{K45}})\ \frac{\Gamma, \rhd\Lambda \Rightarrow \rhd\Delta, A}{\rhd(\Gamma \vee A), \rhd\Lambda \Rightarrow \rhd\Delta, \rhd A}$$

The contingency version of the modal logic $\mathbf{L}$ is built in the language $\mathscr{L}_{\blacktriangleright}$ over $\mathbf{K}$-frames and is denoted as $\mathbf{L}^{\blacktriangleright}$. A semantic condition for contingency operator is presented below:

– $V(\blacktriangleright A, x) = 1$ iff $\exists_{y \in W}(R(x, y)$ and $V(A, y) = 1)$ and $\exists_{y \in W}(R(x, y)$ and $V(A, y) = 0)$.

For $\mathbf{S5}$ we have a simplified version:

– $V(\blacktriangleright A, x) = 1$ iff $\exists_{y \in W} V(A, y) = 1$ and $\exists_{y \in W} V(A, y) = 0$.

Hilbert-style calculi for contingency logics can be obtained from calculi for non-contingency one due to the equalities $\blacktriangleright A = \neg\rhd A$ and $\rhd A = \neg\blacktriangleright A$. For example, Montgomery and Routley [128] give the following system for $\mathbf{T}^{\blacktriangleright}$:

– all schemes of axioms of classical propositional logic,

– $A \to \big(\neg\blacktriangleright(A \to B) \to (\blacktriangleright B \to \blacktriangleright A)\big)$,

– $\blacktriangleright A \leftrightarrow \blacktriangleright\neg A$,

– $\dfrac{A \quad A \to B}{B}$,

– $\dfrac{\vdash A}{\vdash \neg\blacktriangleright A}$.

Similarly, sequent calculi can be obtained from the ones for non-contingency logics.

The essentially and accidentally true versions of the modal logic $\mathbf{L}$, $\mathbf{L}^{\circ}$ and $\mathbf{L}^{\bullet}$, respectively, are built over $\mathbf{L}$-frames in languages $\mathscr{L}_{\circ}$ and $\mathscr{L}_{\bullet}$. The appropriate semantic conditions are as follows:

– $\vartheta(\circ A, x) = 1$ iff $\vartheta(A, x) = 0$ or $\forall_{y \in W}(R(x, y)$ implies $\vartheta(A, y) = 1)$.

– $\vartheta(\bullet A, x) = 1$ iff $\vartheta(A, x) = 1$ and $\exists_{y \in W}(R(x, y)$ and $\vartheta(A, y) = 0)$.

Simplified versions for $\mathbf{S5}^{\circ}$ and $\mathbf{S5}^{\bullet}$ are given below:

– $\vartheta(\circ A, x) = 1$ iff $\vartheta(A, x) = 0$ or $\forall_{y \in W} \vartheta(A, y) = 1$.

– $\vartheta(\bullet A, x) = 1$ iff $\vartheta(A, x) = 1$ and $\exists_{y \in W} \vartheta(A, y) = 0$.

Hilbert-style calculi for the minimal logic of essence and accidence $\mathbf{K}^{\circ\bullet}$ was developed by Marcos [118]. It extends classical propositional logic by the following axioms and rules:

– $(\circ A \wedge \circ B) \to \circ(A \wedge B)$,

– $((A \wedge \circ A)) \vee ((B \wedge \circ B)) \to \circ(A \vee B)$,

– $\bullet A \leftrightarrow \neg\circ A$,

– $\bullet A \to A$,

– $\dfrac{\vdash A}{\vdash \circ A}$,

– $\dfrac{\vdash A \leftrightarrow B}{\vdash \circ A \leftrightarrow \circ B}$.

Fan [41] provides axiomatization of the minimal essence logic $\mathbf{K}^\circ$ (a $\bullet$-free fragment of $\mathbf{K}^{\circ\bullet}$) as an extension of classical propositional logic by the following axioms and rules (we may agree that $\top$ is $p \to p$, for some propositional variable p):

– $\circ\top$,

– $\neg A \to \circ A$,

– $(\circ A \wedge \circ B) \to \circ(A \wedge B)$,

– $\dfrac{\vdash A \to B}{\vdash (\circ A \wedge A) \to \circ B}$.

Fan [41] gives axiomatizations for three extensions of $\mathbf{K}^\circ$:

– $\mathbf{K4}^\circ = \mathbf{K}^\circ + (\circ A \wedge A \to \circ\circ A)$,
– $\mathbf{KB}^\circ = \mathbf{K}^\circ + A \to \circ(\circ\neg A \to \circ A)$,
– $\mathbf{KB5}^\circ = \mathbf{KB}^\circ + \neg\circ\neg A \to \circ(\circ\neg A \to \circ A)$.

For other Hilbert-style axiomatizations of essence and accidence logics see [183, 46].

The essentially and accidentally false versions of the modal logic $\mathbf{L}$, $\mathbf{L}^{\tilde{\circ}}$ and $\mathbf{L}^{\tilde{\bullet}}$, respectively, are built over $\mathbf{L}$-frames in languages $\mathscr{L}_{\tilde{\circ}}$ and $\mathscr{L}_{\tilde{\bullet}}$. We have the following semantic conditions:

– $\vartheta(\tilde{\circ}A, x) = 1$ iff $\vartheta(A, x) = 1$ or $\forall_{y \in W}(R(x, y)$ implies $\vartheta(A, y) = 0)$,

– $\vartheta(\tilde{\bullet}A, x) = 1$ iff $\vartheta(A, x) = 0$ and $\exists_{y \in W}(R(x, y)$ and $\vartheta(A, y) = 1)$.

And their simplified versions for $\mathbf{S5}$-style logics:

– $\vartheta(\tilde{\circ}A, x) = 1$ iff $\vartheta(A, x) = 1$ or $\forall_{y \in W}\vartheta(A, y) = 0$,

– $\vartheta(\tilde{\bullet}A, x) = 1$ iff $\vartheta(A, x) = 0$ and $\exists_{y \in W}\vartheta(A, y) = 1$.

The logic $\mathbf{L}^{\sim}$ is built over $\mathbf{L}$-frames in the language $\mathscr{L}_{\sim}$. The $\neg$-free fragment of $\mathbf{S5}^{\sim}$ is Béziau's [15] paraconsistent logic $\mathbf{Z}$. The logic $\mathbf{L}^{\dot{\sim}}$ is built over $\mathbf{L}$-frames in the language $\mathscr{L}_{\dot{\sim}}$. We introduce a paracomplete companion of $\mathbf{Z}$ as the $\neg$-free fragment of $\mathbf{S5}^{\dot{\sim}}$ and call it $\dot{\mathbf{Z}}$. The semantic conditions for paraconsistent and paracomplete negations are as follows:

– $\vartheta(\sim A, x) = 1$ iff $\exists_{y \in W}(R(x, y)$ and $\vartheta(A, y) = 0)$,

– $\vartheta(\dot{\sim}A, x) = 1$ iff $\forall_{y \in W}(R(x, y)$ implies $\vartheta(A, y) = 0)$.

For $\mathbf{Z}$ and $\dot{\mathbf{Z}}$ simplified version can be provided:

– $\vartheta(\sim A, x) = 1$ iff $\exists_{y \in W}\vartheta(A, y) = 0$,

– $\vartheta(\dot{\sim}A, x) = 1$ iff $\forall_{y \in W}\vartheta(A, y) = 0$.

Hilbert-style calculus of $\mathbf{Z}$ was developed by Béziau [15]. It has all the axioms of the positive fragment of $\mathbf{CPL}$, modus ponens, and the following additional axioms and rules:

- $A \vee {\sim}A$,
- ${\sim}{\sim}A \to A$,
- $\big((A \wedge {\sim}B) \wedge {\sim}(A \wedge {\sim}B)\big) \to (A \wedge {\sim}A)$,
- ${\sim}(A \wedge B) \to ({\sim}A \vee {\sim}B)$,
- $\dfrac{\vdash A \to B}{\vdash {\sim}(A \wedge {\sim}B)}$

Omori and Waragai [139] proved that axioms (1) and (4) were derivable from the other ones. A cut-free hypersequent calculus for **Z** was found by Avron and Lahav [9]: we introduce it in the next section.

Coniglio and Prieto-Sanabria [29] formulated the paraconsistent logic **LTop** (in our terms $\mathbf{S4}^{\sim}$) with topological semantics on the basis of the modal logic **S4**. Hilbert-style calculus for **LTop** [29] consists of the following axioms and rules ($\wedge$ and $\vee$ are defined in a standard way via $\neg$ and $\to$):

- $A \to (B \to A)$,
- $A \to (B \to C) \to ((A \to B) \to (A \to C))$,
- $(\neg A \to B) \to ((\neg A \to \neg B) \to A)$,
- $A \to {\sim}\neg A$,
- ${\sim}\neg{\sim}\neg A \to {\sim}\neg A$,
- ${\sim}(A \wedge B) \to ({\sim}A \vee {\sim}B)$,
- $\neg{\sim}(A \to A)$,
- $\dfrac{A \quad A \to B}{B}$,
- $\dfrac{A \to B}{{\sim}B \to {\sim}A}$,
- $\dfrac{A \quad B}{A \wedge B}$.

Sequent calculi for the logics with negated modalities were formulated by Dodó and Marcos [33] as well as by Lahav, Marcos, and Zohar [109]. The rules for $\sim$ and $\dot{\sim}$ for the case of $\mathbf{K}^{\sim\dot{\sim}}$ were formulated by Dodó and Marcos [33].

$$({\sim}\Rightarrow_{\mathbf{K}})\ \frac{\Gamma \Rightarrow \Delta, A}{{\sim}A, \dot{\sim}\Delta \Rightarrow {\sim}\Gamma} \qquad (\Rightarrow \dot{\sim}_{\mathbf{K}})\ \frac{A, \Gamma \Rightarrow \Delta}{\dot{\sim}\Delta \Rightarrow {\sim}\Gamma, \dot{\sim}A}$$

Lahav, Marcos, and Zohar [109] further developed this approach and found the rules for extensions of $\mathbf{K}^{\sim\dot{\sim}}$: $\mathbf{D}^{\sim\dot{\sim}}$ extends $\mathbf{K}^{\sim\dot{\sim}}$ by $({\sim}\dot{\sim}\mathbf{D})$, $\mathbf{T}^{\sim\dot{\sim}}$ extends $\mathbf{K}^{\sim\dot{\sim}}$ by $(\dot{\sim}\Rightarrow_{\mathbf{T}})$ and $(\Rightarrow {\sim}_{\mathbf{T}})$, $\mathbf{KB}^{\sim\dot{\sim}}$ extends $\mathbf{K}^{\sim\dot{\sim}}$ by $(\Rightarrow \dot{\sim}_{\mathbf{B}})$ and $({\sim}\Rightarrow_{\mathbf{B}})$, $\mathbf{K4}^{\sim\dot{\sim}}$ extends $\mathbf{K}^{\sim\dot{\sim}}$ by $(\Rightarrow \dot{\sim}_{\mathbf{4}})$ and $({\sim}\Rightarrow_{\mathbf{4}})$, $\mathbf{KDB}^{\sim\dot{\sim}}$ extends $\mathbf{KB}^{\sim\dot{\sim}}$ by $({\sim}\dot{\sim}\mathbf{DB})$, $\mathbf{KD4}^{\sim\dot{\sim}}$ extends $\mathbf{K4}^{\sim\dot{\sim}}$ by $({\sim}\dot{\sim}\mathbf{D4})$. As follows from the shape of these rules, both $\sim$ and $\dot{\sim}$ have to be present in the language. Our calculi do not have this restriction. The calculi for $\mathbf{K}^{\sim\dot{\sim}}$, $\mathbf{D}^{\sim\dot{\sim}}$, $\mathbf{T}^{\sim\dot{\sim}}$, $\mathbf{K4}^{\sim\dot{\sim}}$, $\mathbf{KD4}^{\sim\dot{\sim}}$ are shown to enjoy cut admissibility.

$$({\sim}\dot{\sim}\mathbf{D})\ \frac{\Gamma \Rightarrow \Delta}{\dot{\sim}\Delta \Rightarrow {\sim}\Gamma} \qquad (\dot{\sim}\Rightarrow_{\mathbf{T}})\ \frac{\Gamma \Rightarrow \Delta, A}{\dot{\sim}A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow {\sim}_{\mathbf{T}})\ \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, {\sim}A}$$

$$({\sim}\dot{\sim}\mathbf{DB})\ \frac{{\sim}\Gamma, \Pi \Rightarrow \Delta, \dot{\sim}\Lambda}{\Lambda, \dot{\sim}\Delta \Rightarrow \Gamma, {\sim}\Pi} \qquad (\Rightarrow \dot{\sim}_{\mathbf{B}})\ \frac{A, \Gamma, {\sim}\Pi \Rightarrow \Delta, \dot{\sim}\Lambda}{\dot{\sim}\Delta, \Lambda \Rightarrow {\sim}\Gamma, \Pi, \dot{\sim}A} \qquad ({\sim}\Rightarrow_{\mathbf{B}})\ \frac{\Gamma, {\sim}\Pi \Rightarrow \Delta, \dot{\sim}\Lambda, A}{{\sim}A, \dot{\sim}\Delta, \Lambda \Rightarrow {\sim}\Gamma, \Pi}$$

$$({\sim}\dot{\sim}\mathbf{D4})\ \frac{\dot{\sim}\Gamma, \Pi \Rightarrow \Delta, {\sim}\Lambda}{\dot{\sim}\Gamma, \dot{\sim}\Delta \Rightarrow {\sim}\Pi, {\sim}\Lambda} \qquad (\Rightarrow \dot{\sim}_{\mathbf{4}})\ \frac{A, \dot{\sim}\Gamma, \Pi \Rightarrow {\sim}\Delta, \Lambda}{\dot{\sim}\Gamma, \dot{\sim}\Lambda \Rightarrow {\sim}\Delta, {\sim}\Pi, \dot{\sim}A} \qquad ({\sim}\Rightarrow_{\mathbf{4}})\ \frac{\dot{\sim}\Gamma, \Pi \Rightarrow {\sim}\Delta, \Lambda, A}{{\sim}A, \dot{\sim}\Gamma, \dot{\sim}\Lambda \Rightarrow {\sim}\Delta, {\sim}\Pi}$$

2.3 Cut-free hypersequent calculi for S5-style logics with non-standard modalities

An ordered pair written as $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas (of one of the languages considered in this work), is a *sequent*.[8] A finite multiset of sequents written as $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ is a *hypersequent*. Let $\langle W, \vartheta \rangle$ be an **S5**-model. A sequent $\Gamma \Rightarrow \Delta$ is true in a world

[8] This section is significantly based on the material from the author's paper [148].

$w \in W$ iff $\vartheta(A, w) = 0$, for some $A \in \Gamma$, or $\vartheta(B, w) = 1$, for some $B \in \Delta$. A sequent is valid in $\langle W, \vartheta\rangle$ iff it is true in each $w \in W$. A sequent S follows from the set of sequents $\mathscr{S}$ iff for each **S5**-model $\langle W, \vartheta\rangle$, if each $S' \in \mathscr{S}$ is valid $\langle W, \vartheta\rangle$, then S is valid in it as well. A sequent is **S5**-valid iff it is valid in each **S5**-model. A hypersequent H is valid in $\langle W, \vartheta\rangle$ (or $\langle W, \vartheta\rangle$ is a model of H) iff at least one of the components of H is valid in $\langle W, \vartheta\rangle$. A hypersequent H follows from the set of hypersequents $\mathscr{H}$ ($\mathscr{H} \models_{\mathbf{S5}} H$) iff each model of $\mathscr{H}$ is a model of H as well. These notions are defined for the logics with non-standard modalities in a similar way.

Consider Restall's [168] hypersequent calculus $\mathbb{HS}$**S5** for **S5**.[9] It has the following axiom: (Ax) $A \Rightarrow A$. Its structural rules are presented below:

$$(\text{EW}\Rightarrow)\ \frac{H}{A \Rightarrow\ \mid H} \qquad (\Rightarrow\text{EW})\ \frac{H}{\Rightarrow A \mid H} \qquad (\text{IC}\Rightarrow)\ \frac{A, A, \Gamma \Rightarrow \Delta \mid H}{A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\text{IC})\ \frac{\Gamma \Rightarrow \Delta, A, A \mid H}{\Gamma \Rightarrow \Delta, A \mid H}$$

$$(\text{Cut})\ \frac{\Gamma \Rightarrow \Delta, A \mid H \qquad A, \Theta \Rightarrow \Lambda \mid G}{\Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H \mid G} \qquad (\text{Merge})\ \frac{\Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H}{\Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H}$$

In contrast to Restall, we will use a more general version of external weakening, which allows us to add not only a sequent of the form $A \Rightarrow$ or $\Rightarrow A$, but any hypersequent (including empty). The latter issue is important for a constructive cut elimination proof.

$$(\text{EW})\ \frac{G}{G \mid H}$$

One can add internal weakening and external contraction rules:

$$(\text{IW}\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta \mid H}{A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\text{IW})\ \frac{\Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, A \mid H} \qquad (\text{EC})\ \frac{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta \mid H}$$

However, it is not necessary to postulate them as primitive rules:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta \mid H}{\Rightarrow A \mid \Gamma \Rightarrow \Delta \mid H}\,(\text{EW}\Rightarrow)}{A, \Gamma \Rightarrow \Delta \mid H}\,(\text{Merge}) \qquad \dfrac{\dfrac{\Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow A \mid H}\,(\Rightarrow\text{EW})}{\Gamma \Rightarrow \Delta, A \mid H}\,(\text{Merge})$$

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mid H}{\Gamma, \Gamma \Rightarrow \Delta, \Delta \mid H}\,(\text{Merge})}{\Gamma \Rightarrow \Delta \mid H}\,(\text{IC}\Rightarrow), (\Rightarrow\text{IC})$$

The rules for truth-value connectives are as follows:

$$(\neg\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid H}{\neg A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\neg)\ \frac{A, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \neg A \mid H}$$

$$(\wedge\Rightarrow)\ \frac{A, B, \Gamma \Rightarrow \Delta \mid H}{A \wedge B, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow\wedge)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \qquad \Gamma \Rightarrow \Delta, B \mid G}{\Gamma \Rightarrow \Delta, A \wedge B \mid H \mid G}$$

$$(\vee\Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad B, \Gamma \Rightarrow \Delta \mid G}{A \vee B, \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow\vee)\ \frac{\Gamma \Rightarrow \Delta, A, B \mid H}{\Gamma \Rightarrow \Delta, A \vee B \mid H}$$

$$(\to\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \qquad B, \Theta \Rightarrow \Lambda \mid G}{A \to B, \Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H \mid G} \qquad (\Rightarrow\to)\ \frac{A, \Gamma \Rightarrow \Delta, B \mid H}{\Gamma \Rightarrow \Delta, A \to B \mid H}$$

[9] In what follows, we denote via $\mathbb{HS}$**L** a hypersequent calculus for a logic **L**. Similar notation will be used for other types of calculi: $\mathbb{A}$ stands for an axiomatic (Hilbert-style) calculus, $\mathbb{SC}$ stands for an ordinary sequent calculus, $\mathbb{NS}$ stands for a nested sequent calculus, and $\mathbb{ND}$ stands for a natural deduction system.

$$(\leftrightarrow\Rightarrow)\ \frac{B,\Gamma\Rightarrow\Delta,A\mid H\quad A,\Theta\Rightarrow\Lambda,B\mid G}{\Gamma,\Theta\Rightarrow\Delta,\Lambda,A\leftrightarrow B\mid H\mid G}\qquad(\Rightarrow\leftrightarrow)\ \frac{A,B,\Gamma\Rightarrow\Delta\mid H\quad\Theta\Rightarrow\Lambda,A,B\mid G}{A\leftrightarrow B,\Gamma,\Theta\Rightarrow\Delta,\Lambda\mid H\mid G}$$

The rules for necessity and possibility operators are as follows:[10]

$$(\Box\Rightarrow)\ \frac{A,\Gamma\Rightarrow\Delta\mid H}{\Box A\Rightarrow\ \mid\Gamma\Rightarrow\Delta\mid H}\qquad(\Rightarrow\Box)\ \frac{\Rightarrow A\mid H}{\Rightarrow\Box A\mid H}$$

$$(\Diamond\Rightarrow)\ \frac{A\Rightarrow\ \mid H}{\Diamond A\Rightarrow\ \mid H}\qquad(\Rightarrow\Diamond)\ \frac{\Gamma\Rightarrow\Delta,A\mid H}{\Gamma\Rightarrow\Delta\mid\ \Rightarrow\Diamond A\mid H}$$

Let us formulate the rules for the non-standard modalities (all these rules are new except $(\sim\Rightarrow)$ and $(\Rightarrow\sim)$ which were introduced in [9][11]).

$$(\rhd\Rightarrow)\ \frac{A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,A\mid G}{\rhd A\Rightarrow\ \mid\Gamma\Rightarrow\Delta\mid\Theta\Rightarrow\Lambda\mid H\mid G}\qquad(\Rightarrow\rhd)\ \frac{\Rightarrow A\mid A\Rightarrow\ \mid H}{\Rightarrow\ \rhd A\mid H}$$

$$(\blacktriangleright\Rightarrow)\ \frac{\Rightarrow A\mid A\Rightarrow\ \mid H}{\blacktriangleright A\Rightarrow\ \mid H}\qquad(\Rightarrow\blacktriangleright)\ \frac{A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,A\mid G}{\Rightarrow\blacktriangleright A\mid\Gamma\Rightarrow\Delta\mid\Theta\Rightarrow\Lambda\mid H\mid G}$$

$$(\circ\Rightarrow)\ \frac{A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,A\mid G}{\circ A,\Theta\Rightarrow\Lambda\mid\Gamma\Rightarrow\Delta\mid H\mid G}\qquad(\Rightarrow\circ)\ \frac{\Rightarrow A\mid A,\Gamma\Rightarrow\Delta\mid H}{\Gamma\Rightarrow\Delta,\circ A\mid H}$$

$$(\bullet\Rightarrow)\ \frac{\Rightarrow A\mid A,\Gamma\Rightarrow\Delta\mid H}{\bullet A,\Gamma\Rightarrow\Delta\mid H}\qquad(\Rightarrow\bullet)\ \frac{A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,A\mid G}{\Theta\Rightarrow\Lambda,\bullet A\mid\Gamma\Rightarrow\Delta\mid H\mid G}$$

$$(\widetilde{\circ}\Rightarrow)\ \frac{A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,A\mid G}{\widetilde{\circ}A,\Gamma\Rightarrow\Delta\mid\Theta\Rightarrow\Lambda\mid H\mid G}\qquad(\Rightarrow\widetilde{\circ})\ \frac{\Gamma\Rightarrow\Delta,A\mid A\Rightarrow\ \mid H}{\Gamma\Rightarrow\Delta,\widetilde{\circ}A\mid H}$$

$$(\widetilde{\bullet}\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,A\mid A\Rightarrow\ \mid H}{\widetilde{\bullet}A,\Gamma\Rightarrow\Delta\mid H}\qquad(\Rightarrow\widetilde{\bullet})\ \frac{A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,A\mid G}{\Gamma\Rightarrow\Delta,\widetilde{\bullet}A\mid\Theta\Rightarrow\Lambda\mid H\mid G}$$

$$(\sim\Rightarrow)\ \frac{\Rightarrow A\mid H}{\sim A\Rightarrow\ \mid H}\qquad(\Rightarrow\sim)\ \frac{A,\Gamma\Rightarrow\Delta\mid H}{\Gamma\Rightarrow\Delta\mid\ \Rightarrow\sim A\mid H}$$

$$(\dot{\sim}\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,A\mid H}{\dot{\sim}A\Rightarrow\ \mid\Gamma\Rightarrow\Delta\mid H}\qquad(\Rightarrow\dot{\sim})\ \frac{A\Rightarrow\ \mid H}{\Rightarrow\dot{\sim}A\mid H}$$

Let $\clubsuit\in\{\rhd,\blacktriangleright,\circ,\bullet,\widetilde{\circ},\widetilde{\bullet},\sim,\dot{\sim}\}$. A hypersequent calculus $\mathbb{H}\mathbb{S}\mathbf{S5}^{\clubsuit}$ for the logic $\mathbf{S5}^{\clubsuit}$ is obtained from Restall's one for **S5** by the replacement of the rules for $\Box$ and $\Diamond$ with the ones for $\clubsuit$. Hypersequent calculi $\mathbb{H}\mathbb{S}\mathbf{Z}$ and $\mathbb{H}\mathbb{S}\dot{\mathbf{Z}}$, respectively, for logics $\mathbf{Z}$ and $\dot{\mathbf{Z}}$ are $\neg$-free versions of $\mathbb{H}\mathbb{S}\mathbf{S5}^{\sim}$ and $\mathbb{H}\mathbb{S}\mathbf{S5}^{\dot{\sim}}$.

Definition 4 (Proof)**.** By a proof in $\mathbb{H}\mathbb{S}\mathbf{L}$, where $\mathbf{L}$ is one of the logics in question, we mean a tree which nodes are hypersequents such that leaves are axioms and other nodes are obtained from the upper ones by applications of the rules of the calculus.

We write $\mathbb{H}\mathbb{S}\mathbf{L}\vdash H$ iff there is a proof of a hypersequent H in the hypersequent calculus for the logic $\mathbf{L}$. Similarly, $\mathscr{H}\vdash_{\mathbb{H}\mathbb{S}\mathbf{L}} H$ means that there is a proof of a hypersequent H from a finite set of hypersequents $\mathscr{H}$ in $\mathbb{H}\mathbb{S}\mathbf{L}$. If in this proof each cut is on a formula $A\in\Gamma\cup\Delta$ for some component $\Gamma\Rightarrow\Delta$ of some hypersequent in $\mathscr{H}$, then we write $\mathscr{H}\vdash^{\mathrm{cf}}_{\mathbb{H}\mathbb{S}\mathbf{L}} H$. Four examples of proofs in $\mathbb{H}\mathbb{S}\mathbf{S5}^{\rhd}$ are presented in Figure 2.1 (the formulas are taken from Zolin's [200] Hilbert-style axiomatization of $\mathbf{S5}^{\rhd}$).

[10] Restall's original formulation of his calculus [168] does not have the rules for $\vee$, $\rightarrow$, $\leftrightarrow$, and $\Diamond$. These connectives (except $\leftrightarrow$ which we add here) were added to his calculus in [71].

[11] Avron and Lahav's [9] original version of a hypersequent for **Z** is a bit different from the one that we present here. They understand hypersequents as finite sets of sequents which are themselves understood as pairs of finite sets of formulas. They use internal weakening rules, but do not use (Merge) and (IC).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\Rightarrow p, \neg p}\,(\Rightarrow \neg) \qquad \dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}\,(\neg \Rightarrow)}{\rhd p \Rightarrow \mid\ \Rightarrow \neg p \mid \neg p \Rightarrow}\,(\rhd \Rightarrow)}{\rhd p \Rightarrow \mid\ \Rightarrow \rhd \neg p}\,(\Rightarrow \rhd)}{\rhd p \Rightarrow \rhd \neg p}\,(\text{Merge})}{\Rightarrow \rhd p \to \rhd \neg p}\,(\Rightarrow\to)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}\,(\neg \Rightarrow) \qquad \dfrac{p \Rightarrow p}{\Rightarrow p, \neg p}\,(\Rightarrow \neg)}{\rhd \neg p \Rightarrow \mid p \Rightarrow \mid\ \Rightarrow p}\,(\rhd \Rightarrow)}{\rhd \neg p \Rightarrow \mid\ \Rightarrow \rhd p}\,(\Rightarrow \rhd)}{\rhd \neg p \Rightarrow \rhd p}\,(\text{Merge})}{\Rightarrow \rhd \neg p \to \rhd p}\,(\Rightarrow\to) \qquad \dfrac{\dfrac{\dfrac{p \Rightarrow p \qquad p \Rightarrow p}{\rhd p \Rightarrow \mid\ \Rightarrow p \mid p \Rightarrow}\,(\rhd \Rightarrow)}{\rhd p \Rightarrow \mid\ \Rightarrow \rhd p}\,(\Rightarrow \rhd)}{\Rightarrow \rhd\rhd p}\,(\Rightarrow \rhd)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p \qquad q \Rightarrow q}{p, p \to q \Rightarrow q}\,(\to\Rightarrow) \qquad p \Rightarrow p}{\rhd p \Rightarrow \mid p \to q \Rightarrow q \mid p \Rightarrow}\,(\rhd \Rightarrow) \qquad \dfrac{\dfrac{\dfrac{q \Rightarrow q}{p, p, q \Rightarrow q}\,(\text{IW}\Rightarrow)2\text{x}}{p, q \Rightarrow p \to q}\,(\Rightarrow\to) \qquad p \Rightarrow p}{\rhd p \Rightarrow \mid q \Rightarrow p \to q \mid p \Rightarrow}\,(\rhd \Rightarrow)}{\rhd(p \to q) \Rightarrow \mid \rhd p \Rightarrow \mid p \Rightarrow \mid q \Rightarrow \mid\ \Rightarrow q}\,(\rhd \Rightarrow)}{\rhd(p \to q) \Rightarrow \mid \rhd p \Rightarrow \mid p \Rightarrow \mid\ \Rightarrow \rhd q}\,(\Rightarrow \rhd)}{p, \rhd(p \to q), \rhd p \Rightarrow \rhd q}\,(\text{Merge})3\text{x}}{\Rightarrow p \to \big(\rhd(p \to q) \to (\rhd p \to \rhd q)\big)}\,(\Rightarrow\to)3\text{x}$$

Figure 2.1: Examples of proofs in $\mathbb{HS}\mathbf{S5}^{\rhd}$.

2.3.1 Soundness and completeness

Theorem 5 (Strong soundness). *Let $\clubsuit \in \{\rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$ and $\mathbf{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$. For each finite set of hypersequents $\mathscr{H} \cup \{H\}$, if $\mathscr{H} \vdash_{\mathbb{HS}\mathbf{L}} H$, then $\mathscr{H} \models_{\mathbf{L}} H$.*

Proof. Consider the rule $(\Rightarrow \circ)$. Suppose that $\Rightarrow A \mid A, \Gamma \Rightarrow \Delta \mid H$ is valid in an arbitrary $\mathbf{S5}^{\circ}$-model $\langle W, \vartheta \rangle$. Then at least one of the components of this hypersequent is valid in $\langle W, \vartheta \rangle$. If $\Rightarrow A$ is valid in $\langle W, \vartheta \rangle$, then $\vartheta(A, w) = 1$ for all $w \in W$. Then $\Rightarrow \circ A$ is valid in $\langle W, \vartheta \rangle$, and hence $\Gamma \Rightarrow \Delta, \circ A \mid H$ is valid in this model as well. If $A, \Gamma \Rightarrow \Delta$ is valid in $\langle W, \vartheta \rangle$, then $\vartheta(A, w) = 0$ for some $w \in W$ or $\Gamma \Rightarrow \Delta$ is valid in $\langle W, \vartheta \rangle$. Hence, $\Gamma \Rightarrow \Delta, \circ A \mid H$ is valid in $\langle W, \vartheta \rangle$. Obviously, if a component of H is valid in $\langle W, \vartheta \rangle$, then $\Gamma \Rightarrow \Delta, \circ A \mid H$ is valid in it as well.

Consider the rule $(\circ \Rightarrow)$. Suppose that $A, \Gamma \Rightarrow \Delta \mid H$ and $\Theta \Rightarrow \Lambda, A \mid G$ are valid in $\mathbf{S5}^{\circ}$-model $\langle W, \vartheta \rangle$. If H or G is valid in it, then $\circ A, \Theta \Rightarrow \Lambda \mid \Gamma \Rightarrow \Delta \mid H \mid G$ is valid as well. Suppose that $A, \Gamma \Rightarrow \Delta$ is valid in $\langle W, \vartheta \rangle$. Then $\vartheta(A, x) = 0$ for some $x \in W$ or $\Gamma \Rightarrow \Delta$ is valid in $\langle W, \vartheta \rangle$. In the latter case, $\circ A, \Theta \Rightarrow \Lambda \mid \Gamma \Rightarrow \Delta \mid H \mid G$ is valid as well. Let us consider the former one. So suppose that there is $x \in W$ such that $\vartheta(A, x) = 0$. Now assume that $\Theta \Rightarrow \Lambda, A$ is valid in $\langle W, \vartheta \rangle$. Hence, $\Theta \Rightarrow \Lambda$ is valid in $\langle W, \vartheta \rangle$ or $\vartheta(A, y) = 1$ for some $y \in W$. The former disjunct implies the validity of $\circ A, \Theta \Rightarrow \Lambda \mid \Gamma \Rightarrow \Delta \mid H \mid G$. Let us consider the second disjunct. Let $y \in W$ be such that $\vartheta(A, y) = 1$. Since there is $x \in W$ such that $\vartheta(A, x) = 0$, $\vartheta(\circ A, y) = 0$. Therefore, $\circ A, \Theta \Rightarrow \Lambda \mid \Gamma \Rightarrow \Delta \mid H \mid G$ is valid in $\langle W, \vartheta \rangle$.

The other cases are considered similarly. □

Theorem 6 (Strong completeness). *Let $\clubsuit \in \{\rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$ and $\mathbf{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$. For each finite set of hypersequents $\mathscr{H} \cup \{H\}$, if $\mathscr{H} \models_{\mathbf{L}} H$, then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H$.*

Proof. We adapt Avron and Lahav's [9] completeness proof for $\mathbf{Z}$. Suppose that $\mathscr{H} \nvdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H$. We construct a model of $\mathscr{H}$ which is not a model of H. Let $\mathbb{F}$ be the set of subformulas of formulas in $\mathscr{H} \cup \{H\}$. We call a hypersequent G an $\mathbb{F}$-hypersequent iff it satisfies the following conditions:

- if $A \in G$, then $A \in \mathbb{F}$, for each formula A;
- $\mathscr{H} \nvdash^{\mathrm{cf}}_{\mathbb{HSL}} G$;
- if $\Gamma \cup \Delta \subseteq \mathbb{F}$, then either $\Gamma \Rightarrow \Delta \in G$ or $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} G \mid \Gamma \Rightarrow \Delta$.

Since $\mathscr{H}$ is finite, $\mathbb{F}$ is finite as well. Let $S_1, \ldots, S_n$ be an enumeration of all the sequents $\Gamma \Rightarrow \Delta$ such that $\Gamma \cup \Delta \subseteq \mathbb{F}$. We put, for each $1 \leqslant i \leqslant n$:

$$H_0 = H,$$
$$H_i = \begin{cases} H_{i-1} \mid S_i, & \text{if } \mathscr{H} \nvdash^{\mathrm{cf}}_{\mathbb{HSL}} H_{i-1} \mid S_i, \\ H_{i-1} & \text{otherwise}, \end{cases}$$
$$H^* = H_n.$$

Then H^* is an $\mathbb{F}$-hypersequent such that $H \subseteq H^*$. A component $\Gamma^* \Rightarrow \Delta^*$ of H^* is said to be *maximal* iff it has no proper extension in H^* (i.e., if $\Gamma^{**} \Rightarrow \Delta^{**} \in H^*$, $\Gamma^* \subseteq \Gamma^{**}$, and $\Delta^* \subseteq \Delta^{**}$, then $\Gamma^* = \Gamma^{**}$ and $\Delta^* = \Delta^{**}$). Let W be the set of all maximal components of H^*. We write Γ_w and Δ_w (where $w \in W$), respectively, for Γ^* and Δ^* iff $w = \Gamma^* \Rightarrow \Delta^*$. Let ϑ be the valuation such that $\vartheta(p, w) = 1$ iff $p \in \Gamma_w$, for each $p \in \mathcal{P}$.

We need to prove that for each $A \in \mathbb{F}$ and each maximal component w of H^* it holds that:

(a) $A \in \Gamma_w$ implies $\vartheta(A, w) = 1$,

(b) $A \in \Delta_w$ implies $\vartheta(A, w) = 0$.

The proof is by induction on the complexity of A. The basic case (i.e., $A \in \mathcal{P}$) follows from the definition of ϑ. The proof for $\wedge$ and $\sim$ one may find in [9]. Other propositional connectives are considered similarly.

Let A be $\rhd B$. Suppose that $A \in \Gamma_w$. Assume that there is $y \in W$ such that $B \notin \Gamma_y$ and there is a $z \in W$ such that $B \notin \Delta_z$. Since y and z are maximal, $B, \Gamma_y \Rightarrow \Delta_y \notin H^*$ and $\Gamma_z \Rightarrow \Delta_z, B \notin H^*$. Since H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B, \Gamma_y \Rightarrow \Delta_y$ and $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_z \Rightarrow \Delta_z, B$. By the rule ($\rhd \Rightarrow$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \rhd B \Rightarrow \mid \Gamma_y \Rightarrow \Delta_y \mid \Gamma_z \Rightarrow \Delta_z$, i.e., $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid A \Rightarrow \mid \Gamma_y \Rightarrow \Delta_y \mid \Gamma_z \Rightarrow \Delta_z$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid A \Rightarrow \mid y \mid z$. By the rule (Merge), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid A \Rightarrow$. Since $A \in \Gamma_w$, by (Merge) and (IC), we get $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Hence, for each $x \in W$, $B \in \Gamma_x$, or for each $x \in W$, $B \in \Delta_x$. By the induction hypothesis for B, for each $x \in W$, $\vartheta(B, x) = 1$ or for each $x \in W$, $\vartheta(B, x) = 0$. Thus, $\vartheta(A, w) = 1$.

Suppose that $A \in \Delta_w$. Assume that $B \Rightarrow \notin H^*$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B \Rightarrow$, since H^* is an $\mathbb{F}$-sequence. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B \mid B \Rightarrow$. By ($\Rightarrow \rhd$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow \rhd B$, i.e., $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow A$. Since $A \in \Delta_w$, by (Merge) and (IC), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Hence, $B \Rightarrow \in H^*$. Therefore, there is a $y \in W$ such that $B \in \Gamma_y$. By the induction hypothesis for B, there is a $y \in W$ such that $\vartheta(B, y) = 1$. Assume that $\Rightarrow B \notin H^*$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B$, since H^* is an $\mathbb{F}$-sequence. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B \mid B \Rightarrow$, and by ($\Rightarrow \rhd$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow A$ which implies $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Hence, $\Rightarrow B \in H^*$. Thus, there is a $z \in W$ such that $B \in \Delta_z$. By the induction hypothesis for B, there is a $z \in W$ such that $\vartheta(B, z) = 0$. Therefore, $\vartheta(A, w) = 0$.

Let A be $\blacktriangleright B$. Suppose that $A \in \Gamma_w$. We show that there is an $x \in W$ such that $B \in \Delta_x$ and there is an $x \in W$ such that $B \in \Gamma_x$. Assume that $\Rightarrow B \notin H^*$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B$, since H^* is an $\mathbb{F}$-sequence. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B \mid B \Rightarrow$ and by ($\blacktriangleright \Rightarrow$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \blacktriangleright B \Rightarrow$, i.e., $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid A \Rightarrow$ which gives us due to the fact that $A \in \Gamma_w$ and the rules (Merge) and (IC) that $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Thus, $\Rightarrow B \in H^*$. Hence, there is a maximal component x of H^* that extends it, i.e., $B \in \Delta_x$. Therefore, the induction hypothesis for B implies that there is an $x \in W$ such that $\vartheta(B, x) = 0$. Assume that $B \Rightarrow \notin H^*$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B \Rightarrow$, and so that $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B \mid B \Rightarrow$ which implies $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Thus, $B \Rightarrow \in H^*$. Hence, there is a maximal component x of H^* such that $B \in \Gamma_x$. Therefore, the induction hypothesis for B implies that there is an $x \in W$ such that $\vartheta(B, x) = 1$. Thus, $\vartheta(A, x) = 1$.

Suppose that $A \in \Delta_w$. We show that for each $x \in W$, $B \in \Gamma_x$, or for each $x \in W$, $B \in \Delta_x$. Assume the converse, i.e., that there is a $y \in W$ such that $B \notin \Gamma_y$ and there is a $z \in W$ such that $B \notin \Delta_z$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B, \Gamma_y \Rightarrow \Delta_y$ and $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_z \Rightarrow \Delta_z, B$. By ($\Rightarrow \blacktriangleright$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow A \mid \Gamma_y \Rightarrow \Delta_y \mid \Gamma_z \Rightarrow \Delta_z$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow A \mid y \mid z$ which implies $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow A$. Since $A \in \Delta_w$, by (Merge) and (IC), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. By the induction hypothesis for B, for each $x \in W$, $\vartheta(B, x) = 1$ or for each $x \in W$, $\vartheta(B, x) = 0$. Hence, $\vartheta(A, w) = 0$.

Let A be $\circ B$. Suppose that $A \in \Gamma_w$. Suppose that $\{B\} \cup \Delta_w = \emptyset$ and for some maximal $x \in W$, $B \notin \Gamma_x$. Then by the maximality of w and x as well as the fact that H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, B$ and $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B, \Gamma_x \Rightarrow \Delta_x$. By the rule ($\circ \Rightarrow$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \circ B, \Gamma_w \Rightarrow \Delta_w, \mid \Gamma_x \Rightarrow \Delta_x$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid w \mid x$. By (Merge) and (IC), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Thus, $B \in \Delta_w$ or for each $x \in W$, $B \in \Gamma_x$. It follows by the induction hypothesis for B that $\vartheta(B, w) = 0$ or for each maximal $x \in W$, $\vartheta(B, x) = 1$. Hence, $\vartheta(A, w) = 1$.

Suppose that $A \in \Delta_w$. Assume that $B \notin \Gamma_w$ or $\Rightarrow B \notin H^*$. Suppose that $B \notin \Gamma_w$. Then since w is maximal, $B, \Gamma_w \Rightarrow \Delta_w \notin H^*$. Since $B \notin \Gamma_w$ and H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B, \Gamma_w \Rightarrow \Delta_w$. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B \mid B, \Gamma_w \Rightarrow \Delta_w$. By the rule ($\Rightarrow \circ$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \circ B$. Since $A \in \Delta_w$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid w$. Using (Merge) and (IC), we have $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Suppose that $\Rightarrow B \notin H^*$. Since H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B$. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B \mid B, \Gamma_w \Rightarrow \Delta_w$. Using ($\Rightarrow \circ$), (Merge) and (IC), we get $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Hence, $B \in \Gamma_w$ and $\Rightarrow B \in H^*$. Then by the induction hypothesis for B, $\vartheta(B, w) = 1$ and for some $x \in W$, $\vartheta(B, x) = 0$. Hence, $\vartheta(A, w) = 0$.

Let A be $\bullet B$. Suppose that $A \in \Gamma_w$. Assume that $B \notin \Gamma_w$ or $\Rightarrow B \notin H^*$. Suppose that $B \notin \Gamma_w$. Then since w is maximal, $B, \Gamma_w \Rightarrow \Delta_w \notin H^*$. Since $B \notin \Gamma_w$ and H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B, \Gamma_w \Rightarrow \Delta_w$. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B \mid B, \Gamma_w \Rightarrow \Delta_w$. By the rule ($\bullet \Rightarrow$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \bullet B, \Gamma_w \Rightarrow \Delta_w$, i.e., $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid A, \Gamma_w \Rightarrow \Delta_w$. Since $A \in \Gamma_w$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid w$. Using (Merge) and (IC), we have $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Suppose that $\Rightarrow B \notin H^*$. Since H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B$. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B \mid B, \Gamma_w \Rightarrow \Delta_w$. Using ($\bullet \Rightarrow$), (Merge), and (IC), we get $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Hence, $B \in \Gamma_w$ and $\Rightarrow B \in H^*$. Then by the induction hypothesis for B, $\vartheta(B, w) = 1$ and for some $x \in W$ $\vartheta(B, x) = 0$. Hence, $\vartheta(A, w) = 1$.

Suppose that $A \in \Delta_w$. Suppose that $\{B\} \cup \Delta_w = \emptyset$ and for some maximal $x \in W$, $B \notin \Gamma_x$. Then by the maximality of w and x as well as the fact that H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, B$ and $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B, \Gamma_x \Rightarrow \Delta_x$. By the rule ($\Rightarrow \bullet$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \bullet B \mid \Gamma_x \Rightarrow \Delta_x$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid w \mid x$. By (Merge) and (IC), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Thus, $B \in \Delta_w$ or for each maximal $x \in W$, $B \in \Gamma_x$. It follows by the induction hypothesis for B that $\vartheta(B, w) = 0$ or for each maximal $x \in W$, $\vartheta(B, x) = 1$. Hence, $\vartheta(A, w) = 0$.

Let A be $\sim B$. This case is considered in [9] by Avron and Lahav.

The other cases are similar to the previous ones.

Now we show that $\langle W, \vartheta \rangle$ is a model for $\mathscr{H}$, but not for H. Let $H' \in \mathscr{H}$. If its every component is a subsequent of some component of H^*, then due to (EW) and (Merge) H^* is derivable from H' and hence from $\mathscr{H}$ which contradicts to the fact that $\mathscr{H} \nvdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Hence, there is a component $\Gamma \Rightarrow \Delta$ of H' which is not a subsequent of any component of H^*. Let us prove that $\Gamma \Rightarrow \Delta$ is valid in $\langle W, \vartheta \rangle$. Let $w \in W$. Then either $\Gamma \nsubseteq \Gamma_w$ or $\Delta \nsubseteq \Delta_w$. Suppose that $\Gamma \nsubseteq \Gamma_w$ (the case of $\Delta \nsubseteq \Delta_w$ is similar). Then for some $A \in \Gamma$, $A \notin \Gamma_w$. Since $A \in \mathbb{F}$, w is maximal, and H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid A, \Gamma_w \Rightarrow \Delta_w$. Assume that $A \notin \Delta_w$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, A$. By (Cut), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w$, i.e., $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid w$. Hence, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Then $A \in \Delta_w$ which implies, by proposition (b) and the maximality of w, that $\vartheta(A, w) = 0$. Since $A \in \Gamma$, $\Gamma \Rightarrow \Delta$ is true in a world w. Since w is an arbitrary world, $\Gamma \Rightarrow \Delta$ is valid in $\langle W, \vartheta \rangle$ which implies that $\mathscr{H}$ is valid in $\langle W, \vartheta \rangle$ as well.

Assume that $\Gamma \Rightarrow \Delta$ is some component of H. Since $H \subseteq H^*$, there is a maximal component w of H^* such that $\Gamma \subseteq \Gamma_w$ and $\Delta \subseteq \Delta_w$. By propositions (a) and (b), we obtain that $A \in \Gamma$ implies $\vartheta(A, w) = 1$ as well as $A \in \Delta$ implies $\vartheta(A, w) = 0$. Thus, $\Gamma \Rightarrow \Delta$ is not true in a world w. Hence, it is not valid in a model $\langle W, \vartheta \rangle$. Therefore, $\langle W, \vartheta \rangle$ is not a model for H. $\square$

Corollary 7. *Let* $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$ *and* $\mathbf{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$. *For each finite set of hypersequents* $\mathscr{H} \cup \{H\}$, $\mathscr{H} \vdash_{\mathbb{HS}\mathbf{L}} H$ *iff* $\mathscr{H} \models_{\mathbf{L}} H$.

Proof. Follows from Theorems 5 and 6. $\square$

Corollary 8. *Let* $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$, $\mathbf{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$, *and* $\mathscr{H} \cup \{H\}$ *be a finite set of hypersequents. Then* $\mathscr{H} \vdash_{\mathbb{HS}\mathbf{L}} H$ *implies* $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H$.

Proof. Follows from Theorem 6. In the proof of this theorem, (Cut) is used only once to show that $\langle W, \vartheta \rangle$ is a model for $\mathscr{H}$ and is applied only to formulas which belong to $\mathscr{H}$. $\square$

Corollary 9 (Cut admissibility)**.** *Let* $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$, $\mathbf{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$, *and* H *be a hypersequent. Then* $\vdash_{\mathbb{HS}\mathbf{L}} H$ *implies that there is a cut-free proof of* H *in* $\mathbb{HS}\mathbf{L}$.

Proof. Put $\mathscr{H} = \emptyset$ in the proof of Theorem 6. Then the only application of (Cut) in the proof of this Theorem disappears. $\square$

Corollary 10 (Subformula property)**.** *Let* $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$, $\mathbf{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$. *For every hypersequent which is provable in* $\mathbb{HS}\mathbf{L}$ *there is a proof such that each formula which occurs in it is a subformula of the formulas which occur in the conclusion.*

Proof. Follows from Corollary 9 and the fact that in any of the rules of $\mathbb{HS}\mathbf{L}$ each formula which occurs in the premises is a subformula of the formulas which occur in the conclusion. $\square$

Let us recall that strong soundness and completeness as well as cut admissibility theorems are proven for $\mathbf{Z}$ by Avron and Lahav [9] (however, constructive cut admissibility was not proven). Soundness, completeness, and cut admissibility for $\mathbf{S5}$ in the language with $\Box$ were shown by Restall [168] by a Hintikka-style proof, although a bit different from the one which we consider here; he proposed also a sketch of a constructive cut admissibility, a more detailed proof by Metcalfe, Olivetti, and Gabbay's method [123] can be found in Indrzejczak's book [84]. As we know from Corollary 9, in any of the hypersequent calculi in question if we have a proof of a hypersequent H, then we can be sure that there exists a cut-free proof of the same hypersequent. However, the problem is how to find such a cut-free proof. Constructive proof of the cut admissibility theorem will give us an answer.

2.3.2 Constructive proof of the cut admissibility theorem

We use the strategy originally introduced by Metcalfe, Olivetti, and Gabbay [123] for fuzzy logics and further developed by Ciabattoni, Metcalfe, and Montagna [26]. It was adapted for modal logics by Kurokawa [107], Indrzejczak [78, 79, 80, 81, 84], Lellmann [111], Kuznets and Lellmann [108].

Let us recall that a formula introduced by the application of a logical rule is said to be *principal formula*, formulas used for the proof of the principal formula are said to be *side formulas*, all other elements of the hypersequent are said to be *parametric formulas*. We say that a hypersequent which contains the principal formula is an *active hypersequent*.

The length $\mathfrak{l}(\mathfrak{D})$ of a derivation $\mathfrak{D}$ is (the maximal number of applications of inference rules) plus 1 occurring on any branch of $\mathfrak{D}$. The complexity $\mathfrak{c}(A)$ of a formula A is the number of occurrences of its connectives. The cut rank $\mathfrak{r}(\mathfrak{D})$ of a derivation $\mathfrak{D}$ is the maximal complexity of cut formulas in $\mathfrak{D}$ plus 1. Thus, a cut-free derivation $\mathfrak{D}$ has $\mathfrak{r}(\mathfrak{D}) = 0$.

We need to prove two lemmas.

Lemma 11 (Right reduction). *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be derivations such that:*

(1) *$\mathfrak{D}_1$ is a derivation of $\Phi \Rightarrow \Psi, A \mid H$,*

(2) *$\mathfrak{D}_2$ is a derivation of $A^{i_1}, \Upsilon_1 \Rightarrow \Omega_1 \mid \ldots \mid A^{i_n}, \Upsilon_n \Rightarrow \Omega_n \mid G$,*

(3) *$\mathfrak{r}(\mathfrak{D}_1) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_2) \leq \mathfrak{c}(A)$,*

(4) *A is the principal formula of a logical rule in $\mathfrak{D}_1$.*

Then we can construct a derivation $\mathfrak{D}_0$ of $\Phi^{i_1}, \Upsilon_1 \Rightarrow \Omega_1, \Psi^{i_1} \mid \ldots \mid \Phi^{i_n}, \Upsilon_n \Rightarrow \Omega_n, \Psi^{i_n} \mid H \mid G$ such that $\mathfrak{r}(\mathfrak{D}_0) \leq \mathfrak{c}(A)$.

Proof. The proof is by induction on on $\mathfrak{l}(\mathfrak{D}_2)$. Basic case is easy and is omitted here.

Inductive case. We have different cases depending on the last rule applied to $\mathfrak{D}_2$.

Case 1. The last rule is applied on only side sequents G. The case is obvious and is omitted.

Case 2. The last rule is any non-modal rule that does not have A as the principal formula.

Let us use the following abbreviations, for any $1 \leq l \leq n$ and $1 \leq k \leq n$, where $x, y \in \{l, i_l, j_l\}$:

- $\mathfrak{A}_l^{\Phi\Psi} = \Phi_l \Rightarrow \Psi_l$,
- $\mathfrak{A}_{i_l}^{\Phi\Psi} = \Phi^{i_l} \Rightarrow \Psi^{i_l}$,
- $\mathfrak{A}_l^{\Phi\Psi} \times \mathfrak{A}_k^{\Upsilon\Omega} = \Phi_l, \Upsilon_k \Rightarrow \Psi_l, \Omega_k$.
- $\mathfrak{A}_{i_l}^{\Phi\Psi} \times \mathfrak{A}_k^{\Upsilon\Omega} = \Phi^{i_l}, \Upsilon_k \Rightarrow \Psi^{i_l}, \Omega_k$.

Subcase 2.1. The rule of the last inference of $\mathfrak{D}_2$ is (Merge).

$$\frac{A^{i_1}, \Gamma_1 \Rightarrow \Delta_1 \mid A^{i_2}, \Gamma_2 \Rightarrow \Delta_2 \mid A^{i_3}, \mathfrak{A}_3^{\Gamma\Delta} \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Gamma\Delta} \mid H}{A^{i_1+i_2}, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \mid A^{i_3}, \mathfrak{A}_3^{\Gamma\Delta} \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Gamma\Delta} \mid H}$$

$\mathfrak{D}_1$ ends with the hypersequent $\Theta \Rightarrow \Lambda, A \mid G$. What we need to obtain is

$$\Gamma_1, \Gamma_2, \Theta^{i_1+i_2} \Rightarrow \Delta_1, \Delta_2, \Lambda^{i_1+i_2} \mid \mathfrak{A}_3^{\Gamma\Delta} \times \mathfrak{A}_{i_3}^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta} \times \mathfrak{A}_{i_n}^{\Theta\Lambda} \mid H \mid G$$

Using the induction hypothesis and (Merge), we get the required result as follows:

$$\frac{\Gamma_1, \Theta^{i_1} \Rightarrow \Delta_1, \Lambda^{i_1} \mid \Gamma_2, \Theta^{i_2} \Rightarrow \Delta_2, \Lambda^{i_2} \mid \mathfrak{A}_3^{\Gamma\Delta} \times \mathfrak{A}_{i_3}^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta} \times \mathfrak{A}_{i_n}^{\Theta\Lambda} \mid H \mid G}{\Gamma_1, \Gamma_2, \Theta^{i_1+i_2} \Rightarrow \Delta_1, \Delta_2, \Lambda^{i_1+i_2} \mid \mathfrak{A}_3^{\Gamma\Delta} \times \mathfrak{A}_{i_3}^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Gamma\Delta} \times \mathfrak{A}_{i_n}^{\Theta\Lambda} \mid H \mid G}$$

The other cases are considered similarly.

Case 3. The last inference is an application of the non-modal left introduction rule whose principal formula is A. This case is rather easy and is omitted here, since it deals with classical propositional logic for which cut admissibility is well-known (some propositional cases can be found in [84]).

Case 4. The rule of the last inference of $\mathfrak{D}_2$ is $(\rhd \Rightarrow)$.

Subcase 4.1. A is principal in $\mathfrak{D}_2$ and $A = \rhd B$. The last inference of $\mathfrak{D}_2$ looks as follows.

$$\frac{B, \rhd B^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \rhd B^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid G_1 \qquad \rhd B^{i_1}, \mathfrak{A}_1^{\Pi\Sigma}, B \mid \ldots \mid \rhd B^{i_n}, \mathfrak{A}_n^{\Pi\Sigma} \mid G_2}{\rhd B \Rightarrow \mid \rhd B^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \rhd B^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid \rhd B^{i_1}, \mathfrak{A}_1^{\Pi\Sigma} \mid \ldots \mid \rhd B^{i_n}, \mathfrak{A}_n^{\Pi\Sigma} \mid G_1 \mid G_2}$$

Since $\mathfrak{D}_1$ ends as the condition (4) states, the last inference of $\mathfrak{D}_1$ is as follows.

$$\frac{\Rightarrow B \mid B \Rightarrow \mid H}{\Rightarrow \rhd B \mid H}$$

What we need to obtain is

$$\Rightarrow \mid \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid \mathfrak{A}_1^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_1 \mid G_2$$

By the induction hypothesis, we obtain derivations $\mathfrak{D}_3$ and $\mathfrak{D}_4$, respectively, of the following hypersequents such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_4) \leq \mathfrak{c}(A)$:

$$B, \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_2^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G_1.$$

$$\mathfrak{A}_1^{\Pi\Sigma}, B \mid \mathfrak{A}_2^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_2.$$

Using these hypersequents and $\Rightarrow B \mid B \Rightarrow \mid H$, by (Cut), (Merge) with (IC) (or just (EC)) as well as (EW), we get

$$\dfrac{\dfrac{\dfrac{\mathfrak{A}_1^{\Pi\Sigma}, B \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_2 \qquad \dfrac{\Rightarrow B \mid B \Rightarrow \mid H \qquad B, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G_1}{B \Rightarrow \mid \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid H \mid G_1}}{\mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid \mathfrak{A}_1^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid H \mid H \mid G_1 \mid G_2}}{\mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid \mathfrak{A}_1^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_1 \mid G_2}}{\Rightarrow \mid \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid \mathfrak{A}_1^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_1 \mid G_2}$$

Subcase 4.2. The rule of the last inference of $\mathfrak{D}_2$ is $(\rhd \Rightarrow)$ and the principal formula in $\mathfrak{D}_2$ is not A. Then the last inference of $\mathfrak{D}_2$ looks as follows.

$$\dfrac{B, A^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid G_1 \qquad A^{i_1}, \mathfrak{A}_1^{\Pi\Sigma}, B \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Pi\Sigma} \mid G_2}{\rhd B \Rightarrow \mid A^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid A^{i_1}, \mathfrak{A}_1^{\Pi\Sigma} \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Pi\Sigma} \mid G_1 \mid G_2}$$

$\mathfrak{D}_1$ ends with the hypersequent $\Gamma \Rightarrow \Delta, A \mid H$. We should obtain

$$\rhd B \Rightarrow \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_1 \mid G_2$$

By the induction hypothesis, we obtain derivations $\mathfrak{D}_3$ and $\mathfrak{D}_4$, respectively, of the following hypersequents such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_4) \leq \mathfrak{c}(A)$:

$$B, \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G_1.$$

$$\mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Pi\Sigma}, B \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Pi\Sigma} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Pi\Sigma} \mid H \mid G_2.$$

Applying $(\rhd \Rightarrow)$ and structural rules, we get the required result.

In what follows, we omit the cases where A is not the principal formula in $\mathfrak{D}_2$, since they are similar to the Subcase 4.2.

Case 5. The rule of the last inference of $\mathfrak{D}_2$ is $(\blacktriangleright \Rightarrow)$. Hence, $A = \blacktriangleright B$ and A is principal in $\mathfrak{D}_2$. $\mathfrak{D}_2$ ends as follows:

$$\dfrac{\Rightarrow B \mid B \Rightarrow \mid \blacktriangleright B^{i_1}, \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid \blacktriangleright B^{i_n}, \Pi_n \Rightarrow \Sigma_n \mid G}{\blacktriangleright B \Rightarrow \mid \blacktriangleright B^{i_1}, \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid \blacktriangleright B^{i_n}, \Pi_n \Rightarrow \Sigma_n \mid G}$$

$\mathfrak{D}_1$ ends as follows:

$$\dfrac{B, \Gamma \Rightarrow \Delta \mid H_1 \qquad \Theta \Rightarrow \Lambda, B \mid H_2}{\Rightarrow \blacktriangleright B \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H_1 \mid H_2}$$

We should obtain

$$\Rightarrow \mid \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H_1 \mid H_2 \mid G$$

By the induction hypothesis, we have

$$\Rightarrow B \mid B \Rightarrow \mid \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H_1 \mid H_2 \mid G$$

Using the induction hypothesis, applying (Cut) on the formulas of lower complexity, and using other structural rules, we obtain the required result as follows:

$$\dfrac{\dfrac{\dfrac{\Theta \Rightarrow \Lambda, B \mid H_2 \qquad \dfrac{\Rightarrow B \mid B \Rightarrow \mid \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H_1 \mid H_2 \mid G \qquad B, \Gamma \Rightarrow \Delta \mid H_1}{\Gamma \Rightarrow \Delta \mid B \Rightarrow \mid \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H_1 \mid H_1 \mid H_2 \mid G}}{\Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H_1 \mid H_1 \mid H_2 \mid H_2 \mid G}}{\Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H_1 \mid H_2 \mid G}}{\Rightarrow \mid \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H_1 \mid H_2 \mid G}$$

Case 6. The rule of the last inference of $\mathfrak{D}_2$ is $(\sim\Rightarrow)$. Hence, $A = {\sim}B$ and A is principal in $\mathfrak{D}_2$. $\mathfrak{D}_2$ ends as follows:

$$\dfrac{\Rightarrow B \mid {\sim}B^{i_1}, \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid {\sim}B^{i_n}, \Pi_n \Rightarrow \Sigma_n \mid G}{{\sim}B \Rightarrow \mid {\sim}B^{i_1}, \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid {\sim}B^{i_n}, \Pi_n \Rightarrow \Sigma_n \mid G}$$

$\mathfrak{D}_1$ ends as follows:

$$\dfrac{B, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow {\sim}B \mid H}$$

We should obtain

$$\Gamma \Rightarrow \Delta \mid \Rightarrow \mid \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n \mid H \mid G$$

Using the induction hypothesis, applying (Cut) on the formulas of lower complexity, and using other structural rules, we obtain the required result as follows:

$$\dfrac{\Gamma \Rightarrow \Delta \mid \Rightarrow B \mid \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n \mid H \mid G \qquad B, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \mid \Pi_1 \Rightarrow \Sigma_1 \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n \mid H \mid G}$$

Case 7. The rule of the last inference of $\mathfrak{D}_2$ is $(\dot{\sim}\Rightarrow)$. A is principal in $\mathfrak{D}_2$ and $A = \dot{\sim}B$. The last step of $\mathfrak{D}_2$ is as follows.

$$\dfrac{\dot{\sim}B^{i_1}, \mathfrak{A}_1^{\Theta\Lambda}, B \mid \dot{\sim}B^{i_2}, \mathfrak{A}_2^{\Theta\Lambda} \mid \ldots \mid \dot{\sim}B^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid G}{\dot{\sim}B \Rightarrow \mid \dot{\sim}B^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \dot{\sim}B^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid G}$$

The last step of $\mathfrak{D}_1$ is as follows:

$$\dfrac{B \Rightarrow \mid H}{\Rightarrow \dot{\sim}B \mid H}$$

We should obtain

$$\Rightarrow \mid \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G$$

Using the induction hypothesis, applying (Cut) on the formulas of lower complexity, and using other structural rules, we obtain the required result as follows:

$$\dfrac{\dfrac{\dfrac{\mathfrak{A}_1^{\Theta\Lambda}, B \mid \mathfrak{A}_2^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G \qquad B \Rightarrow \mid H}{\mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid H \mid G}}{\mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G}}{\Rightarrow \mid \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G}$$

Case 8. The rule of the last inference of $\mathfrak{D}_2$ is $(\circ\Rightarrow)$. A is principal in $\mathfrak{D}_2$ and $A = \circ B$. The last inference of $\mathfrak{D}_2$ looks as follows.

$$\dfrac{B, \circ B^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \circ B^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid G_1 \qquad \circ B^{i_1}, \mathfrak{A}_1^{\Pi\Sigma}, B \mid \ldots \mid \circ B^{i_n}, \mathfrak{A}_n^{\Pi\Sigma} \mid G_2}{\circ B^{i_1+1}, \mathfrak{A}_1^{\Pi\Sigma} \mid \circ B^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \circ B^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid \circ B^{i_2}, \mathfrak{A}_2^{\Pi\Sigma} \mid \ldots \mid \circ B^{i_n}, \mathfrak{A}_n^{\Pi\Sigma} \mid G_1 \mid G_2}$$

Since $\mathfrak{D}_1$ ends as the condition (4) states, the last inference of $\mathfrak{D}_1$ is as follows.

$$\dfrac{\Rightarrow B \mid B, \Gamma \Rightarrow \Delta \ \mid H}{\Gamma \Rightarrow \Delta, \circ B \mid H}$$

We should obtain

$$\mathfrak{A}^{\Gamma\Delta}_{i_1+1} \times \mathfrak{A}^{\Pi\Sigma}_1 \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Theta\Lambda}_n \mid \mathfrak{A}^{\Gamma\Delta}_{i_2} \times \mathfrak{A}^{\Pi\Sigma}_2 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Pi\Sigma}_n \mid H \mid G_1 \mid G_2$$

By the induction hypothesis, we obtain derivations $\mathfrak{D}_3$ and $\mathfrak{D}_4$, respectively, of the following hypersequents such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_4) \leq \mathfrak{c}(A)$:

$$B, \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \mathfrak{A}^{\Gamma\Delta}_{i_2} \times \mathfrak{A}^{\Theta\Lambda}_2 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Theta\Lambda}_n \mid H \mid G_1$$

$$\mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Pi\Sigma}_1, B \mid \mathfrak{A}^{\Gamma\Delta}_{i_2} \times \mathfrak{A}^{\Pi\Sigma}_2 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Pi\Sigma}_n \mid H \mid G_2$$

Let us abbreviate them as follows:

$$B, \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \mathfrak{A}_1 \mid H$$

$$\mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Pi\Sigma}_1, B \mid \mathfrak{A}_2 \mid H$$

Then we reason as follows, using (Cut) and (EC):

$$\dfrac{\dfrac{\mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Pi\Sigma}_1, B \mid \mathfrak{A}_2 \mid H \qquad \dfrac{\Rightarrow B \mid B, \Gamma \Rightarrow \Delta \;\mid H \qquad B, \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \mathfrak{A}_1 \mid H}{B, \Gamma \Rightarrow \Delta \;\mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \mathfrak{A}_1 \mid H \mid H}}{\mathfrak{A}^{\Gamma\Delta}_{i_1+1} \times \mathfrak{A}^{\Pi\Sigma}_1 \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \mathfrak{A}_1 \mid \mathfrak{A}_2 \mid H \mid H \mid H}}{\mathfrak{A}^{\Gamma\Delta}_{i_1+1} \times \mathfrak{A}^{\Pi\Sigma}_1 \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \mathfrak{A}_1 \mid \mathfrak{A}_2 \mid H}$$

The cases dealing with $\bullet$, $\widetilde{\circ}$, and $\widetilde{\bullet}$ are considered similarly. ☐

Lemma 12 (Left reduction). *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be derivations such that:*

(1) $\mathfrak{D}_1$ *is a derivation of* $\Phi_1 \Rightarrow \Psi_1, A^{i_1} \mid \ldots \mid \Phi_n \Rightarrow \Psi_n, A^{i_n} \mid H$,

(2) $\mathfrak{D}_2$ *is a derivation of* $A, \Upsilon \Rightarrow \Omega \mid G$,

(3) $\mathfrak{r}(\mathfrak{D}_1) \leq \mathfrak{c}(A)$ *and* $\mathfrak{r}(\mathfrak{D}_2) \leq \mathfrak{c}(A)$.

Then we can construct a derivation $\mathfrak{D}_0$ of $\Phi_1, \Upsilon^{i_1} \Rightarrow \Omega^{i_1}, \Psi_1 \mid \ldots \mid \Phi_n, \Upsilon^{i_n} \Rightarrow \Omega^{i_n}, \Psi_n \mid H \mid G$ such that $\mathfrak{r}(\mathfrak{D}_0) \leq \mathfrak{c}(A)$.

Proof. The proof is by induction on $\mathfrak{l}(\mathfrak{D}_1)$. The basic case is easy and left for the reader.

Inductive case. We have different cases depending on the last rule applied to $\mathfrak{D}_1$. The first three cases are similar to Lemma 11.

Case 4. The rule of the last inference of $\mathfrak{D}_1$ is $(\rhd \Rightarrow)$. In this case, A is not the principal formula. The last inference of $\mathfrak{D}_1$ is as follows.

$$\dfrac{B, \mathfrak{A}^{\Theta\Lambda}_1, A^{i_1} \mid \ldots \mid \mathfrak{A}^{\Theta\Lambda}_n, A^{i_n} \mid G_1 \qquad \mathfrak{A}^{\Pi\Sigma}_1, A^{i_1}, B \mid \ldots \mid \mathfrak{A}^{\Pi\Sigma}_n, A^{i_n} \mid G_2}{\rhd B \Rightarrow \mid \mathfrak{A}^{\Theta\Lambda}_1, A^{i_1} \mid \ldots \mid \mathfrak{A}^{\Theta\Lambda}_n, A^{i_n} \mid \mathfrak{A}^{\Pi\Sigma}_1, A^{i_1} \mid \ldots \mid \mathfrak{A}^{\Pi\Sigma}_n, A^{i_n} \mid G_1 \mid G_2}$$

$\mathfrak{D}_2$ ends with the hypersequent $A, \Gamma \Rightarrow \Delta \mid H$. We should obtain:

$$\rhd B \Rightarrow \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Theta\Lambda}_n \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Pi\Sigma}_1 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Pi\Sigma}_n \mid H \mid G_1 \mid G_2$$

By the induction hypothesis, we obtain derivations $\mathfrak{D}_3$ and $\mathfrak{D}_4$, respectively, of the following hypersequents such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_4) \leq \mathfrak{c}(A)$:

$$B, \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \mathfrak{A}^{\Gamma\Delta}_{i_2} \times \mathfrak{A}^{\Theta\Lambda}_2 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Theta\Lambda}_n \mid H \mid G_1.$$

$$\mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1, B \mid \mathfrak{A}^{\Gamma\Delta}_{i_2} \times \mathfrak{A}^{\Theta\Lambda}_2 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Theta\Lambda}_n \mid H \mid G_2.$$

Applying $(\rhd \Rightarrow)$, we get the required result.

Case 5. The rule of the last inference of $\mathfrak{D}_1$ is $(\Rightarrow \rhd)$. Subcase 5.1. $A = \rhd B$ and is the principal formula. $\mathfrak{D}_1$ ends as follows:

$$\frac{B \Rightarrow | \Rightarrow B \mid \Pi_1 \Rightarrow \Sigma_1, \triangleright B^{i_1} \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n, \triangleright B^{i_n} \mid H}{\Rightarrow \triangleright B \mid \Pi_1 \Rightarrow \Sigma_1, \triangleright B^{i_1} \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n, \triangleright B^{i_n} \mid H}$$

$\mathfrak{D}_2$ ends as follows: $\triangleright B, \Gamma \Rightarrow \Delta \mid G$.

By Lemma 11, the claim holds since this case satisfies the condition of application of the Lemma.

Subcase 5.2. A is not the principal formula. The last inference of $\mathfrak{D}_1$ is as follows.

$$\frac{B \Rightarrow \mid \Rightarrow B \mid \Pi_1 \Rightarrow \Sigma_1, A^{i_1} \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n, A^{i_n} \mid G}{\Rightarrow \triangleright B \mid \Pi_1 \Rightarrow \Sigma_1, A^{i_1} \mid \ldots \mid \Pi_n \Rightarrow \Sigma_n, A^{i_n} \mid G}$$

$\mathfrak{D}_2$ ends with the hypersequent $A, \Gamma \Rightarrow \Delta \mid H$. We should obtain:

$$\Rightarrow \triangleright B \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Pi\Sigma}_1 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Pi\Sigma}_n \mid H \mid G$$

By the induction hypothesis and $(\Rightarrow \triangleright)$, we get the required result similarly to Case 4 of this Lemma.

Case 6. The rule of the last inference of $\mathfrak{D}_1$ is $(\blacktriangleright \Rightarrow)$. In this case A is not principal and is contained in a side hypersequent G. The case is similar to the Case 1 of Lemma 11.

Case 7. The rule of the last inference of $\mathfrak{D}_1$ is $(\Rightarrow \blacktriangleright)$.

Subcase 7.1. A is the principal formula. The last inference of $\mathfrak{D}_1$ is as follows.

$$\frac{B, \mathfrak{A}^{\Theta\Lambda}_1, \blacktriangleright B^{i_1} \mid \ldots \mid \mathfrak{A}^{\Theta\Lambda}_n, \blacktriangleright B^{i_n} \mid G_1 \qquad \mathfrak{A}^{\Pi\Sigma}_1, \blacktriangleright B^{i_1}, B \mid \ldots \mid \mathfrak{A}^{\Pi\Sigma}_n, \blacktriangleright B^{i_n} \mid G_2}{\Rightarrow \blacktriangleright B \mid \mathfrak{A}^{\Theta\Lambda}_1, \blacktriangleright B^{i_1} \mid \ldots \mid \mathfrak{A}^{\Theta\Lambda}_n, \blacktriangleright B^{i_n} \mid \mathfrak{A}^{\Pi\Sigma}_1, \blacktriangleright B^{i_1} \mid \ldots \mid \mathfrak{A}^{\Pi\Sigma}_n, \blacktriangleright B^{i_n} \mid G_1 \mid G_2}$$

$\mathfrak{D}_2$ ends with the hypersequent $\blacktriangleright B, \Gamma \Rightarrow \Delta \mid H$. We should obtain

$$\Gamma \Rightarrow \Delta \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Theta\Lambda}_n \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Pi\Sigma}_1 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Pi\Sigma}_n \mid H \mid G_1 \mid G_2$$

Using the inductive hypothesis and $(\Rightarrow \blacktriangleright)$, we get the following inference.

$$\frac{\Gamma \Rightarrow \Delta \mid B, \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \mathfrak{A}^{\Gamma\Delta\Theta\Lambda}_{2,\ldots,n} \mid H \mid G_1 \qquad \Gamma \Rightarrow \Delta \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Pi\Sigma}_1, B \mid \mathfrak{A}^{\Gamma\Delta\Pi\Sigma}_{2,\ldots,n} \mid H \mid G_2}{\Gamma \Rightarrow \Delta \mid \Rightarrow \blacktriangleright B \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \mathfrak{A}^{\Gamma\Delta\Theta\Lambda}_{2,\ldots,n} \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Pi\Sigma}_1 \mid \mathfrak{A}^{\Gamma\Delta\Pi\Sigma}_{2,\ldots,n} \mid H \mid G_1 \mid G_2}$$

where

- $\mathfrak{A}^{\Gamma\Delta\Theta\Lambda}_{2,\ldots,n} = \mathfrak{A}^{\Gamma\Delta}_{i_2} \times \mathfrak{A}^{\Theta\Lambda}_2 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Theta\Lambda}_n$
- $\mathfrak{A}^{\Gamma\Delta\Pi\Sigma}_{2,\ldots,n} = \mathfrak{A}^{\Gamma\Delta}_{i_2} \times \mathfrak{A}^{\Pi\Sigma}_2 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Pi\Sigma}_n$

By Lemma 11, the claim holds since this case satisfies the condition of application of the Lemma.

Subcase 7.2. A is not the principal formula. The last inference of $\mathfrak{D}_1$ is as follows.

$$\frac{B, \mathfrak{A}^{\Theta\Lambda}_1, A^{i_1} \mid \ldots \mid \mathfrak{A}^{\Theta\Lambda}_n, A^{i_n} \mid G_1 \qquad \mathfrak{A}^{\Pi\Sigma}_1, A^{i_1}, B \mid \ldots \mid \mathfrak{A}^{\Pi\Sigma}_n, A^{i_n} \mid G_2}{\Rightarrow \blacktriangleright B \mid \mathfrak{A}^{\Theta\Lambda}_1, A^{i_1} \mid \ldots \mid \mathfrak{A}^{\Theta\Lambda}_n, A^{i_n} \mid \mathfrak{A}^{\Pi\Sigma}_1, A^{i_1} \mid \ldots \mid \mathfrak{A}^{\Pi\Sigma}_n, A^{i_n} \mid G_1 \mid G_2}$$

$\mathfrak{D}_2$ ends with the hypersequent $A, \Gamma \Rightarrow \Delta \mid H$. We should obtain:

$$\Rightarrow \blacktriangleright B \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Theta\Lambda}_1 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Theta\Lambda}_n \mid \mathfrak{A}^{\Gamma\Delta}_{i_1} \times \mathfrak{A}^{\Pi\Sigma}_1 \mid \ldots \mid \mathfrak{A}^{\Gamma\Delta}_{i_n} \times \mathfrak{A}^{\Pi\Sigma}_n \mid H \mid G_1 \mid G_2$$

By the induction hypothesis and $(\blacktriangleright \Rightarrow)$, we get the required result similarly to Case 4 of this Lemma.

Case 8. The rule of the last inference of $\mathfrak{D}_1$ is $(\sim \Rightarrow)$. In this case A is not principal and is contained in a side hypersequent G. The case is similar to the Case 1 of Lemma 11.

The other cases are treated similarly. $\square$

Theorem 13 (Constructive elimination of cuts). *Let* $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$ *and* $\mathbf{L} \in \{\mathbf{S5}^{\clubsuit}, \mathbf{Z}, \dot{\mathbf{Z}}\}$. *If a derivation* $\mathfrak{D}$ *in* $\mathbb{H}\mathbb{S}\mathbf{L}$ *has an application of* (Cut), *then it can be transformed into a cut-free derivation* $\mathfrak{D}'$.

Proof. Assume that a derivation $\mathfrak{D}$ in $\mathbb{HS}\mathbf{L}$ has at least one application of (Cut), i.e., $\mathfrak{r}(\mathfrak{D}) > 0$. The proof proceeds by the double induction on $\langle \mathfrak{r}(\mathfrak{D}), \mathfrak{nr}(\mathfrak{D})\rangle$, where $\mathfrak{nr}(\mathfrak{D})$ is the number of applications of (Cut) in $\mathfrak{D}$. Consider the uppermost application of (Cut) in $\mathfrak{D}$ with a cut rank $\mathfrak{r}(\mathfrak{D})$. We apply Lemma 12 to its premises and decrease either $\mathfrak{r}(\mathfrak{D})$ or $\mathfrak{nr}(\mathfrak{D})$. Then we can use the inductive hypothesis. □

2.3.3 A few more modalities

There are other modalities one may also consider. For example, Pan and Yang [143] introduced the following weak essentially true and strong accidentally true modalities:

- $\vartheta(\circledast A, x) = 1$ iff $\vartheta(A, x) = 0$ or $\exists_{y\in W}\vartheta(A, y) = 1$,
- $\vartheta(\odot A, x) = 1$ iff $\vartheta(A, x) = 1$ and $\forall_{y\in W}\vartheta(A, y) = 0$.

Thus, $\circledast A = \neg A \vee \Diamond A = A \to \Diamond A$ and $\odot A = A \wedge \Box\neg A$. Since these modalities are quite unusual, we decided not to include them into the main part of our paper, but we can present sound, complete, and cut-free hypersequent calculi for them:

$$(\circledast\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \quad A \Rightarrow\ \mid G}{\circledast A, \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow\circledast)\ \frac{A, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda, A \mid H}{\Gamma \Rightarrow \Delta, \circledast A \mid \Theta \Rightarrow \Lambda \mid H}$$

$$(\odot\Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda, A \mid H}{\odot A, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H} \qquad (\Rightarrow\odot)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \quad A \Rightarrow\ \mid G}{\Gamma \Rightarrow \Delta, \odot A \mid H \mid G}$$

By analogy we may define weak essentially false and strong accidentally false modalities as follows:

- $\vartheta(\widetilde{\circledast} A, x) = 1$ iff $\vartheta(A, x) = 1$ or $\exists_{y\in W}\vartheta(A, y) = 0$,
- $\vartheta(\widetilde{\odot} A, x) = 1$ iff $\vartheta(A, x) = 0$ and $\forall_{y\in W}\vartheta(A, y) = 1$.

Hence, $\widetilde{\circledast} A = A \vee \Diamond\neg A = \neg A \to \Diamond\neg A$ and $\widetilde{\odot} A = \neg A \wedge \Box A$. The appropriate sound, complete, and cut-free hypersequent calculi for them are presented below:

$$(\widetilde{\circledast}\Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \quad \Rightarrow A \mid G}{\widetilde{\circledast} A, \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow\widetilde{\circledast})\ \frac{\Gamma \Rightarrow \Delta, A \mid A, \Theta \Rightarrow \Lambda \mid H}{\Gamma \Rightarrow \Delta, \widetilde{\circledast} A \mid \Theta \Rightarrow \Lambda \mid H}$$

$$(\widetilde{\odot}\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid A, \Theta \Rightarrow \Lambda \mid H}{\widetilde{\odot} A, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H} \qquad (\Rightarrow\widetilde{\odot})\ \frac{A, \Gamma \Rightarrow \Delta \mid H \quad \Rightarrow A \mid G}{\Gamma \Rightarrow \Delta, \widetilde{\odot} A \mid H \mid G}$$

Among other non-standard modalities we would like to mention the so-called 'boxdot' modality $\boxdot A = \Box A \wedge A$ introduced by Boolos [18] for the needs of provability logic and being interpreted as 'provable and true' (for its use in the context of essence and accident logics see [185]):

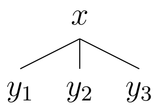
- $\vartheta(\boxdot A, x) = 1$ iff $\vartheta(A, x) = 1$ and $\forall_{y\in W}\vartheta(A, y) = 1$.

The appropriate rules for $\boxdot$ are as follows:

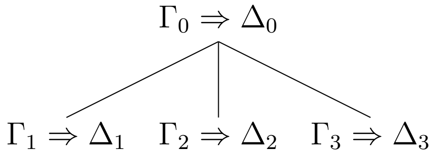
$$(\boxdot\Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad A, \Theta \Rightarrow \Lambda \mid G}{\boxdot A, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H \mid G} \qquad (\Rightarrow\boxdot)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \quad \Rightarrow A \mid G}{\Gamma \Rightarrow \Delta, \boxdot A \mid H \mid G}$$

2.4 Cut-free nested sequent calculi for the logics with non-standard modalities weaker than S5

For the case of the logics weaker than **S5** we need a more general approach than hypersequent calculus.[12] Such an approach is a generalisation of hypersequents which was suggested independently by various authors under various names: nested sequents (Kashima, 1994, [94]), deep sequents (Brünnler, 2006, [22]), and tree-hypersequents (Poggiolesi, 2008 [158, 157]). We will follow Poggiolesi's explication of this method. To begin with, let us give an informal explanation of the notion of a nested sequent. Let us consider the following Kripke tree:



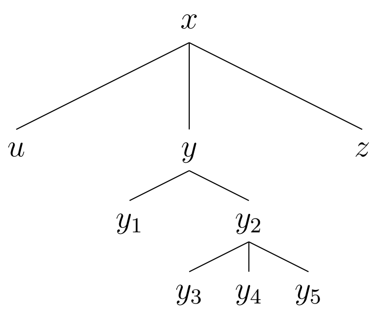
To present this structure proof-theoretically we replace worlds with sequents:



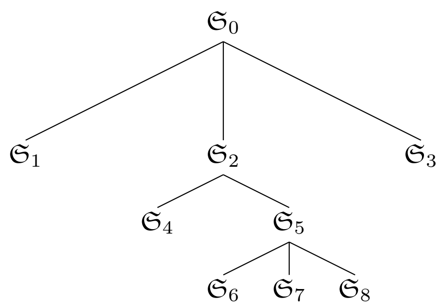
Of course, if we formulate rules with such notation, they will be too clumsy. So it is better to use a more compact notation:

$$\Gamma_0 \Rightarrow \Delta_0 / \Gamma_1 \Rightarrow \Delta_1; \Gamma_2 \Rightarrow \Delta_2; \Gamma_3 \Rightarrow \Delta_3$$

We can think about more complicated example:



We put $\mathfrak{S}_i = \Gamma_i \Rightarrow \Delta_i$ (where $0 \leqslant i \leqslant 8$). Then we obtain the following picture:



And it can be written as follows:

$$\mathfrak{S}_0 / \mathfrak{S}_1; (\mathfrak{S}_2 / \mathfrak{S}_4; (\mathfrak{S}_5 / \mathfrak{S}_6; \mathfrak{S}_7; \mathfrak{S}_8)); \mathfrak{S}_3$$

[12] This section contains previously unpublished results by the author.

To sum up, a nested sequent is an encoding of Kripke trees by means of sequents. Now let us give a formal definition, following [158, 157].

Definition 14 (Nested sequent)**.** [157, Definition 6.1]

- If $\mathfrak{S}$ is a sequent, then $\mathfrak{S}$ is a nested sequent.
- If $\mathfrak{S}$ is a sequent and $\mathfrak{N}_1, \ldots, \mathfrak{N}_m$ are nested sequents, then $\mathfrak{S}/\mathfrak{N}_1; \ldots; \mathfrak{N}_m$ is a nested sequent.

Definition 15. A translation τ of a sequent $\Gamma \Rightarrow \Delta$ into a formula is defined as follows, where $A_1, \ldots, A_l$ are all the elements of Γ, and $B_1, \ldots, B_m$ are all the elements of Δ:

$$\tau(\Gamma \Rightarrow \Delta) = \begin{cases} (A_1 \wedge \ldots \wedge A_l) \to (B_1 \vee \ldots \vee B_m) & \text{iff } \Gamma \neq \emptyset, \Delta \neq \emptyset; \\ B_1 \vee \ldots \vee B_m & \text{iff } \Gamma = \emptyset, \Delta \neq \emptyset; \\ \neg(A_1 \wedge \ldots \wedge A_l) & \text{iff } \Gamma \neq \emptyset, \Delta = \emptyset; \\ p \wedge \neg p & \text{iff } \Gamma = \emptyset, \Delta = \emptyset. \end{cases}$$

Definition 16. [157, Definition 6.2] A translation σ of a nested sequent built into a formula is defined as follows:

- $\rho(\Gamma \Rightarrow \Delta) = \tau(\Gamma \Rightarrow \Delta)$;
- $\rho(\mathfrak{S}/\mathfrak{N}_1; \ldots; \mathfrak{N}_m) = \rho(\mathfrak{S}) \vee \Box\rho(\mathfrak{N}_1) \vee \ldots \vee \Box\rho(\mathfrak{N}_m)$.

In the case of logics without $\Box$ in the language, one may express it via other connectives and modify this definition in the appropriate way. For example, if $\rhd$ is in the language, then the second clause of this definition is as follows:

- $\rho(\mathfrak{S}/\mathfrak{N}_1; \ldots; \mathfrak{N}_m) = \rho(\mathfrak{S}) \vee \big(\rho(\mathfrak{N}_1) \wedge \rhd\rho(\mathfrak{N}_1)\big) \vee \ldots \vee \big(\rho(\mathfrak{N}_m) \wedge \rhd\rho(\mathfrak{N}_m)\big)$.

Of course, this translation works only for logics with non-standard modalities such that $\Box$ is expressed in them (only reflexive (non)contingency logics and only serial essence (accidence) logics). However, later we will show a more general way of checking the soundness of the rules that is applicable to all the logics in question.

Let $\mathfrak{N}$ be a nested sequent and $\mathfrak{S}$ be a sequent which is a part of $\mathfrak{N}$. Then we write $\mathfrak{N}[\mathfrak{S}]$, if we want to say something about $\mathfrak{S}$. A formal treatment of the expression $\mathfrak{N}[\mathfrak{S}]$ is given in [157], where the notion of a zoom tree-hypersequent is introduced.

Definition 17. [157, Definition 6.3] The notion of a zoom nested sequent is inductively defined as follows:

- $[*]$ is a zoom nested sequent,
- if $\mathfrak{N}_1, \ldots, \mathfrak{N}_l$ are nested sequents, then $[*]/\mathfrak{N}_1; \ldots; \mathfrak{N}_l$ is a zoom nested sequent,
- if $\mathfrak{N}_1[*]$ is a zoom nested sequent and $\mathfrak{N}_2, \ldots, \mathfrak{N}_l$ are nested sequents, then $[*]/\mathfrak{N}_1[*]; \ldots; \mathfrak{N}_l$ is a zoom nested sequent,
- if $\mathfrak{S}$ is a sequent, $\mathfrak{N}_1[*]$ is a zoom nested sequent, and $\mathfrak{N}_2, \ldots, \mathfrak{N}_l$ are nested sequents, then $\mathfrak{S}/\mathfrak{N}_1[*]; \ldots; \mathfrak{N}_l$ is a zoom nested sequent,
- if $\mathfrak{S}$ is a sequent, $\mathfrak{N}_1[*][*]$ is a zoom nested sequent, $\mathfrak{N}_2, \ldots, \mathfrak{N}_l$ are nested sequents, then $\mathfrak{S}/\mathfrak{N}_1[*][*]; \ldots; \mathfrak{N}_l$ is a zoom nested sequent.

Definition 18. [157, Definition 6.4] For all zoom nested sequents $\mathfrak{N}[*]$, or $\mathfrak{N}[*][*]$, and nested sequents $\mathfrak{K}$ and $\mathfrak{L}$, we define $\mathfrak{N}[\mathfrak{K}]$ and $\mathfrak{N}[\mathfrak{K}][\mathfrak{L}]$, the result of substituting $\mathfrak{K}$ into $\mathfrak{N}[*]$, and the result of substituting $\mathfrak{K}$ and $\mathfrak{L}$ in $\mathfrak{N}[*][*]$, respectively, as follows, where $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are sequents:

- if $\mathfrak{N}[*] = [*]$, then $\mathfrak{N}[\mathfrak{K}] = \mathfrak{K}$,

- if $\mathfrak{N}[*] = [*]/\mathfrak{N}_1;\ldots;\mathfrak{N}_l$ and $\mathfrak{K} = \mathfrak{S}_1/\mathfrak{M}_1;\ldots;\mathfrak{M}_m$, then $\mathfrak{N}[\mathfrak{K}] = \mathfrak{S}_1/\mathfrak{N}_1;\ldots;\mathfrak{N}_l;\mathfrak{M}_1;\ldots;\mathfrak{M}_m$,
- if $\mathfrak{N}[*][*] = [*]/\mathfrak{N}_1[*];\ldots;\mathfrak{N}_l$ and $\mathfrak{K} = \mathfrak{S}_1/\mathfrak{M}_1;\ldots;\mathfrak{M}_m$, then $\mathfrak{N}[\mathfrak{K}][\mathfrak{L}] = \mathfrak{S}_1/\mathfrak{N}_1[\mathfrak{L}];\ldots;\mathfrak{N}_l;\mathfrak{M}_1;\ldots;\mathfrak{M}_m$,
- if $\mathfrak{N}[*] = \mathfrak{S}_2/\mathfrak{N}_1[*],\ldots,\mathfrak{N}_l$, then $\mathfrak{N}[\mathfrak{K}] = \mathfrak{S}_2/\mathfrak{N}_1[\mathfrak{K}],\ldots,\mathfrak{N}_l$,
- if $\mathfrak{N}[*][*] = \mathfrak{S}_2/\mathfrak{N}_1[*][*],\ldots,\mathfrak{N}_l$, then $\mathfrak{N}[\mathfrak{K}][\mathfrak{L}] = \mathfrak{S}_2/\mathfrak{N}_1[\mathfrak{K}][\mathfrak{L}],\ldots,\mathfrak{N}_l$.

Let us describe Poggiolesi's nested sequent (tree-hypersequent) calculi for modal logics [157, Section 6.2, p. 126–127]. The axiom (applied for any propositional variable p) and propositional rules are as follows:

$$\mathfrak{N}[p,\Gamma \Rightarrow \Delta, p]$$

$$[\neg\Rightarrow]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A]}{\mathfrak{N}[\neg A,\Gamma \Rightarrow \Delta]} \qquad [\Rightarrow\neg]\ \frac{\mathfrak{N}[A,\Gamma \Rightarrow \Delta]}{\mathfrak{N}[\Gamma \Rightarrow \Delta,\neg A]}$$

$$[\wedge\Rightarrow]\ \frac{\mathfrak{N}[A,B,\Gamma \Rightarrow \Delta]}{\mathfrak{N}[A\wedge B,\Gamma \Rightarrow \Delta]} \qquad [\Rightarrow\wedge]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta, B]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A\wedge B]}$$

$$[\vee\Rightarrow]\ \frac{\mathfrak{N}[A,\Gamma \Rightarrow \Delta] \quad \mathfrak{N}[B,\Gamma \Rightarrow \Delta]}{\mathfrak{N}[A\vee B,\Gamma \Rightarrow \Delta]} \qquad [\Rightarrow\vee]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A, B]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A\vee B]}$$

$$[\to\Rightarrow]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A] \quad \mathfrak{N}[B,\Gamma \Rightarrow \Delta]}{\mathfrak{N}[A\to B,\Gamma \Rightarrow \Delta]} \qquad [\Rightarrow\to]\ \frac{\mathfrak{N}[A,\Gamma \Rightarrow \Delta, B]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A\to B]}$$

$$[\leftrightarrow\Rightarrow]\ \frac{\mathfrak{N}[B,\Gamma \Rightarrow \Delta, A] \quad \mathfrak{N}[A,\Gamma \Rightarrow \Delta, B]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A\leftrightarrow B]} \qquad [\Rightarrow\leftrightarrow]\ \frac{\mathfrak{N}[A,B,\Gamma \Rightarrow \Delta] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta, A, B]}{\mathfrak{N}[A\leftrightarrow B,\Gamma \Rightarrow \Delta]}$$

Modal rules for the logic **K** (where X is a multiset of nested sequents) are given below:

$$[\Box\Rightarrow]\ \frac{\mathfrak{N}[\Box A,\Gamma \Rightarrow \Delta/(A,\Theta \Rightarrow \Lambda/X)]}{\mathfrak{N}[\Box A,\Gamma \Rightarrow \Delta/(\Theta \Rightarrow \Lambda/X)]} \qquad [\Rightarrow\Box]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/\Rightarrow A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta,\Box A]}$$

Poggiolesi [157] does not formulate rules for $\Diamond$, but it is quite easy to find such rules, using the equality $\Diamond A = \neg\Box\neg A$:

$$[\Diamond\Rightarrow]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/A\Rightarrow]}{\mathfrak{N}[\Diamond A,\Gamma \Rightarrow \Delta]} \qquad [\Rightarrow\Diamond]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta,\Diamond A/(\Theta \Rightarrow \Lambda, A/X)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta,\Diamond A/(\Theta \Rightarrow \Lambda/X)]}$$

As for extensions of **K**, Poggiolesi proposes special logical rules (in two versions, see pp. 125 and 127 in [157], we present the second one) and special structural rules ([157, p. 125]). Each pair of these rules corresponds to axioms/properties of the accessibility relation. Both special logical and structural rules are sound with respect to frames with the corresponding properties of their accessibility relations. **T**-, **D**-, **B**-, **4**-, and **5**-axioms can be proved by means of the corresponding special logical rules as well as the corresponding structural logical rules. Both types of the rules are helpful for the cut elimination: in order to eliminate cuts produced by special logical rules, one needs special structural rules. At that special structural rules are shown to be admissible in the calculi with special logical rules [157, Lemmas 6.13–6.17]. On the other hand, a careful examination of Poggiolesi's cut elimination proof shows that if we consider calculi without special logical rules, but with special structural rules postulated as primitive ones, then, obviously, we do not have cases produced by special logical rules (which are the most complicated ones), but we have new cases produced by special structural rules which can be easily solved (such cases are observed and treated in our cut elimination proof for non-standard modalities in Section 2.4.2). At that as Poggiolesi notes, the rules corresponding to **5**-axiom “do not reflect the strength and power“ of this axiom [157, p. 126], since there are some problems with cut elimination (the calculus for **K5** obtained in such a way is not cut-free) and even completeness. As a result, in our study of non-standard modalities, we do not treat Euclidean logics, in order to avoid such problems. Let us list the above-mentioned special logical and structural rules:

$$[\mathbf{D}]\ \frac{\mathfrak{N}[\Box A, \Gamma \Rightarrow \Delta / A \Rightarrow]}{\mathfrak{N}[\Box A, \Gamma \Rightarrow \Delta]} \qquad [\widetilde{\mathbf{D}}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \Rightarrow]}{\mathfrak{N}[\Gamma \Rightarrow \Delta]}$$

$$[\mathbf{T}]\ \frac{\mathfrak{N}[\Box A, A, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[\Box A, \Gamma \Rightarrow \Delta]} \qquad [\widetilde{\mathbf{T}}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / X)]}{\mathfrak{N}[\Gamma, \Theta \Rightarrow \Delta, \Lambda / X]}$$

$$[\mathbf{4}]\ \frac{\mathfrak{N}[\Box A, \Gamma \Rightarrow \Delta / (\Box A, \Theta \Rightarrow \Lambda / X)]}{\mathfrak{N}[\Box A, \Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / X)]} \qquad [\widetilde{\mathbf{4}}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / X)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Rightarrow / \Theta \Rightarrow \Lambda / X)]}$$

$$[\mathbf{B}]\ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta / (\Box A, \Theta \Rightarrow \Lambda / X)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / X)]} \qquad [\widetilde{\mathbf{B}}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / (\Xi \Rightarrow \Pi / X); Y)]}{\mathfrak{N}[\Gamma, \Xi \Rightarrow \Delta, \Pi / (\Theta \Rightarrow \Lambda / X; Y)]}$$

$$[\mathbf{5}]\ \frac{\mathfrak{N}[\Box A, \Gamma \Rightarrow \Delta / (\Box A, \Theta \Rightarrow \Lambda / X)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Box A, \Theta \Rightarrow \Lambda / X)]} \qquad [\widetilde{\mathbf{5}}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / (\Xi \Rightarrow \Pi / X); Y)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Xi \Rightarrow \Pi / X); (\Theta \Rightarrow \Lambda / Y)]}$$

As follows from [157, Lemma 10.6], all the propositional rules, the modal rules and the special logical rules are invertible. The following structural rules are shown to be height-preserving admissible [157, Lemmas 10.2–10.5, 10.7]:

$$[\mathrm{EW}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta]}{\mathfrak{N}[\Gamma \Rightarrow \Delta / \Pi \Rightarrow \Sigma]} \qquad [\mathrm{IW}{\Rightarrow}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta]}{\mathfrak{N}[A, \Gamma \Rightarrow \Delta]} \qquad [{\Rightarrow}\mathrm{IW}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A]}$$

$$[\mathrm{Merge}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Pi \Rightarrow \Sigma / X); (\Theta \Rightarrow \Lambda / Y)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta / (\Pi, \Theta \Rightarrow \Sigma, \Lambda / X; Y)]}$$

$$[\mathrm{rn}]\ \frac{\mathfrak{N}}{\Rightarrow / \mathfrak{N}} \qquad [\mathrm{C}{\Rightarrow}]\ \frac{\mathfrak{N}[A, A, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[A, \Gamma \Rightarrow \Delta]} \qquad [{\Rightarrow}\mathrm{C}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A, A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A]}$$

In order to formulate cut in nested sequent approach, we need some auxiliary definitions.

Definition 19. [157, Definition 6.5] Given two nested sequents, $\mathfrak{N}[\Gamma \Rightarrow \Delta]$ and $\mathfrak{M}[\Theta \Rightarrow \Lambda]$ together with an occurrence of a sequent in each, the relation of an equivalent position between two of their sequents, in this case $\Gamma \Rightarrow \Delta$ and $\Theta \Rightarrow \Lambda$, $\mathfrak{N}[\Gamma \Rightarrow \Delta] \approx \mathfrak{M}[\Theta \Rightarrow \Lambda]$, is defined inductively in the following way:

- $\Gamma \Rightarrow \Delta \approx \Theta \Rightarrow \Lambda$
- $\Gamma \Rightarrow \Delta / X \approx \Theta \Rightarrow \Lambda / Y$
- If $\mathfrak{K}[\Gamma \Rightarrow \Delta] \approx \mathfrak{L}[\Theta \Rightarrow \Lambda]$, then $\Phi \Rightarrow \Pi / \mathfrak{K}[\Gamma \Rightarrow \Delta]; X \approx \Sigma \Rightarrow \Upsilon / \mathfrak{L}[\Theta \Rightarrow \Lambda]; Y$.

“Intuitively, given two nested sequents, $\mathfrak{N}[\Gamma \Rightarrow \Delta]$ and $\mathfrak{M}[\Theta \Rightarrow \Lambda]$ together with an occurrence of a sequent in each, the relation of equivalent position between two of their sequents holds when, by considering $\mathfrak{N}[\Gamma \Rightarrow \Delta]$ and $\mathfrak{M}[\Theta \Rightarrow \Lambda]$ as trees, and $\Gamma \Rightarrow \Delta$ and $\Theta \Rightarrow \Lambda$ as nodes of the trees, the two nodes have the same height in their respective trees.” [158, p. 36] [the notation and terminology adjusted].

Definition 20. [157, Definition 6.6] Given two nested sequents $\mathfrak{N}[\Gamma \Rightarrow \Delta]$ and $\mathfrak{M}[\Theta \Rightarrow \Lambda]$ together with an occurrence of a sequent in each, such that $\mathfrak{N}[\Gamma \Rightarrow \Delta] \approx \mathfrak{M}[\Theta \Rightarrow \Lambda]$, the operation of product, $\mathfrak{N}[\Gamma \Rightarrow \Delta] \otimes \mathfrak{M}[\Theta \Rightarrow \Lambda]$, is defined inductively in the following way:

- $\Gamma \Rightarrow \Delta \otimes \Theta \Rightarrow \Lambda = \Gamma, \Theta \Rightarrow \Delta, \Lambda$
- $(\Gamma \Rightarrow \Delta/X) \otimes (\Theta \Rightarrow \Lambda/Y) = \Gamma, \Theta \Rightarrow \Delta, \Lambda/X; Y$
- $(\Phi \Rightarrow \Pi/\mathfrak{K}[\Gamma \Rightarrow \Delta]; X) \otimes (\Psi \Rightarrow \Upsilon/\mathfrak{L}[\Theta \Rightarrow \Lambda]; Y) = \Phi, \Psi \Rightarrow \Pi, \Upsilon/(\mathfrak{K}[\Gamma \Rightarrow \Delta] \otimes \mathfrak{L}[\Theta \Rightarrow \Lambda]); X; Y.$

Given two tree-hypersequents $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ and $\mathfrak{M}[A, \Theta \Rightarrow \Lambda]$ together with an occurrence of a sequent in each, such that $\mathfrak{N}[\Gamma \Rightarrow \Delta, A] \approx \mathfrak{M}[A, \Theta \Rightarrow \Lambda]$, the cut rule is:

$$[\mathrm{Cut}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A] \quad \mathfrak{M}[A, \Theta \Rightarrow \Lambda]}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Theta \Rightarrow \Delta, \Lambda]}$$

Let $\clubsuit \in \{\rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$. Let us formulate a nested sequent calculus $\mathbb{NSK}^{\clubsuit}$ for the logic $\mathbf{K}^{\clubsuit}$. It has propositional rules and the following modal rules:

$$[\rhd \Rightarrow]\ \frac{\mathfrak{N}[\rhd A, \Gamma \Rightarrow \Delta/(A, \Theta \Rightarrow \Lambda/X)] \quad \mathfrak{N}[\rhd A, \Gamma \Rightarrow \Delta/(\Xi \Rightarrow \Pi, A/Y)]}{\mathfrak{N}[\rhd A, \Gamma \Rightarrow \Delta/(\Theta \Rightarrow \Lambda/X); (\Xi \Rightarrow \Pi/Y)]}$$

$$[\Rightarrow \rhd_L]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/A \Rightarrow]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \rhd A]} \qquad [\Rightarrow \rhd_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/ \Rightarrow A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \rhd A]}$$

$$[\blacktriangleright \Rightarrow_L]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/A \Rightarrow]}{\mathfrak{N}[\blacktriangleright A, \Gamma \Rightarrow \Delta]} \qquad [\blacktriangleright \Rightarrow_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/ \Rightarrow A]}{\mathfrak{N}[\blacktriangleright A, \Gamma \Rightarrow \Delta]}$$

$$[\Rightarrow \blacktriangleright]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacktriangleright A/(A, \Theta \Rightarrow \Lambda/X)] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta, \blacktriangleright A/(\Xi \Rightarrow \Pi, A/Y)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \blacktriangleright A/(\Theta \Rightarrow \Lambda/X); (\Xi \Rightarrow \Pi/Y)]}$$

$$[\circ \Rightarrow]\ \frac{\mathfrak{N}[\circ A, \Gamma \Rightarrow \Delta/(A, \Theta \Rightarrow \Lambda/X)] \quad \mathfrak{N}[\circ A, \Xi \Rightarrow \Pi, A/Y]}{\mathfrak{N}[\circ A, \Gamma, \Xi \Rightarrow \Delta, \Pi/Y; (\Theta \Rightarrow \Lambda/X)]}$$

$$[\Rightarrow \circ_L]\ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \circ A]} \qquad [\Rightarrow \circ_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/ \Rightarrow A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \circ A]}$$

$$[\bullet \Rightarrow_L]\ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[\bullet A, \Gamma \Rightarrow \Delta]} \qquad [\bullet \Rightarrow_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/ \Rightarrow A]}{\mathfrak{N}[\bullet A, \Gamma \Rightarrow \Delta]}$$

$$[\Rightarrow \bullet]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \bullet A/(A, \Theta \Rightarrow \Lambda/X)] \quad \mathfrak{N}[\Xi \Rightarrow \Pi, A, \bullet A/Y]}{\mathfrak{N}[\Gamma, \Xi \Rightarrow \Delta, \Pi, \bullet A/Y; (\Theta \Rightarrow \Lambda/X)]}$$

$$[\widetilde{\circ} \Rightarrow]\ \frac{\mathfrak{N}[\widetilde{\circ} A, A, \Theta \Rightarrow \Lambda/X] \quad \mathfrak{N}[\widetilde{\circ} A, \Gamma \Rightarrow \Delta/(\Xi \Rightarrow \Pi, A/Y)]}{\mathfrak{N}[\widetilde{\circ} A, \Gamma, \Theta \Rightarrow \Delta, \Lambda/X; (\Xi \Rightarrow \Pi/Y)]}$$

$$[\Rightarrow \widetilde{\circ}_L]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/A \Rightarrow]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \widetilde{\circ} A]} \qquad [\Rightarrow \widetilde{\circ}_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \widetilde{\circ} A]}$$

$$[\widetilde{\bullet} \Rightarrow_L]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/A \Rightarrow]}{\mathfrak{N}[\widetilde{\bullet} A, \Gamma \Rightarrow \Delta]} \qquad [\widetilde{\bullet} \Rightarrow_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A]}{\mathfrak{N}[\widetilde{\bullet} A, \Gamma \Rightarrow \Delta]}$$

$$[\Rightarrow \widetilde{\bullet}]\ \frac{\mathfrak{N}[A, \Theta \Rightarrow \Lambda, \widetilde{\bullet} A/X] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta/(\Xi \Rightarrow \Pi, A, \widetilde{\bullet} A/Y)]}{\mathfrak{N}[\Gamma, \Theta \Rightarrow \Delta, \Lambda, \widetilde{\bullet} A/X; (\Xi \Rightarrow \Pi/Y)]}$$

$$[\sim\Rightarrow]\ \dfrac{\mathfrak{N}[\Gamma\Rightarrow\Delta/\Rightarrow A]}{\mathfrak{N}[\sim A,\Gamma\Rightarrow\Delta]}\qquad [\Rightarrow\sim]\ \dfrac{\mathfrak{N}[\Gamma\Rightarrow\Delta,\sim A/(A,\Theta\Rightarrow\Lambda/X)]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,\sim A/(\Theta\Rightarrow\Lambda/X)]}$$

$$[\dot\sim\Rightarrow]\ \dfrac{\mathfrak{N}[\dot\sim A,\Gamma\Rightarrow\Delta/(\Theta\Rightarrow\Lambda,A/X)]}{\mathfrak{N}[\dot\sim A,\Gamma\Rightarrow\Delta/(\Theta\Rightarrow\Lambda/X)]}\qquad [\Rightarrow\dot\sim]\ \dfrac{\mathfrak{N}[\Gamma\Rightarrow\Delta/A\Rightarrow]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,\dot\sim A]}$$

Let $\clubsuit\in\{\rhd,\blacktriangleright,\circ,\bullet,\widetilde{\circ},\widetilde{\bullet},\sim,\dot\sim\}$ and $\mathbf{X}_1,\dots\mathbf{X}_m\in\{\mathbf{T},\mathbf{D},\mathbf{4},\mathbf{B}\}$. A nested sequent calculus $\mathbb{N}\$\mathbf{KX}_1,\dots\mathbf{X}_m^{\clubsuit}$ for a logic $\mathbf{KX}_1,\dots\mathbf{X}_m^{\clubsuit}$ is an extension of $\mathbb{N}\$\mathbf{K}^{\clubsuit}$ by the rules $[\widetilde{\mathbf{X}_1}],\dots,[\widetilde{\mathbf{X}_m}]$.

Lemma 21. *Let $\clubsuit\in\{\rhd,\blacktriangleright,\circ,\bullet,\widetilde{\circ},\widetilde{\bullet},\sim,\dot\sim\}$ and $\mathbf{X}_1,\dots\mathbf{X}_m\in\{\mathbf{T},\mathbf{D},\mathbf{4},\mathbf{B}\}$. All the propositional rules and the modal rules of nested sequent calculi $\mathbb{N}\$\mathbf{K}^{\clubsuit}$ and $\mathbb{N}\$\mathbf{KX}_1,\dots\mathbf{X}_m^{\clubsuit}$ are invertible.*

Proof. Similarly to [157, Lemma 10.6]. □

Lemma 22. *Let $\clubsuit\in\{\rhd,\blacktriangleright,\circ,\bullet,\widetilde{\circ},\widetilde{\bullet},\sim,\dot\sim\}$ and $\mathbf{X}_1,\dots\mathbf{X}_m\in\{\mathbf{T},\mathbf{D},\mathbf{4},\mathbf{B}\}$. The rules* [EW], [IW⇒], [⇒IW], [Merge], [rn], [C⇒], *and* [⇒C] *are height-preserving admissible in $\mathbb{N}\$\mathbf{K}^{\clubsuit}$ and $\mathbb{N}\$\mathbf{KX}_1,\dots\mathbf{X}_m^{\clubsuit}$.*

Proof. Similarly to Lemmas 10.2–10.5 and 10.7 from [157]. □

Let us present some examples of proofs. Some proofs in $\mathbb{N}\$\mathbf{K}^{\rhd}$ are given below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\rhd A\Rightarrow /A\Rightarrow A\qquad \rhd A\Rightarrow /A\Rightarrow A}{\rhd A\Rightarrow /\Rightarrow A;A\Rightarrow}\,[\rhd\Rightarrow]}{\rhd A\Rightarrow /\neg A\Rightarrow;\Rightarrow\neg A}\,[\Rightarrow\neg],[\neg\Rightarrow]}{\rhd A\Rightarrow\rhd\neg A/\Rightarrow\neg A}\,[\Rightarrow\rhd_L]}{\rhd A\Rightarrow\rhd\neg A,\rhd\neg A}\,[\Rightarrow\rhd_R]}{\rhd A\Rightarrow\rhd\neg A}\,[\Rightarrow C]}{\Rightarrow\rhd A\to\rhd\neg A}\,[\Rightarrow\to]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\rhd\neg A\Rightarrow /A\Rightarrow A}{\rhd\neg A\Rightarrow /\neg A,A\Rightarrow}\,[\neg\Rightarrow]\qquad \dfrac{\rhd\neg A\Rightarrow /A\Rightarrow A}{\rhd\neg A\Rightarrow /\Rightarrow A,\neg A}\,[\Rightarrow\neg]}{\rhd\neg A\Rightarrow /A\Rightarrow;\Rightarrow A}\,[\rhd\Rightarrow]}{\rhd\neg A\Rightarrow\rhd A/A\Rightarrow}\,[\Rightarrow\rhd_L]}{\rhd\neg A\Rightarrow\rhd A,\rhd A}\,[\Rightarrow\rhd_R]}{\rhd\neg A\Rightarrow\rhd A}\,[\Rightarrow C]}{\Rightarrow\rhd\neg A\to\rhd A}\,[\Rightarrow\to]$$

$$\dfrac{\dfrac{\Rightarrow A}{\Rightarrow /\Rightarrow A}\,[\mathrm{rn}]}{\Rightarrow\rhd A}\,[\Rightarrow\rhd_R]$$

An example of a proof in $\mathbb{N}\$\mathbf{T}^{\rhd}$, where $\mathfrak{X}$ is $\rhd A,\rhd(A\to B)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\rhd\Rightarrow]\ \dfrac{\dfrac{\mathfrak{X}\Rightarrow /A\Rightarrow A\qquad \mathfrak{X}\Rightarrow /B\Rightarrow B}{\mathfrak{X}\Rightarrow /A,A\to B\Rightarrow B}\qquad \mathfrak{X}\Rightarrow /A\Rightarrow A}{\mathfrak{X}\Rightarrow /A\to B\Rightarrow B;A\Rightarrow}\qquad [\rhd\Rightarrow]\ \dfrac{\dfrac{\dfrac{\mathfrak{X}\Rightarrow /B\Rightarrow B}{\mathfrak{X}\Rightarrow /A,A,B\Rightarrow B}}{\mathfrak{X}\Rightarrow /A,B\Rightarrow A\to B}\qquad \mathfrak{X}\Rightarrow /A\Rightarrow A}{\mathfrak{X}\Rightarrow /B\Rightarrow A\to B;A\Rightarrow}}{\rhd(A\to B),\rhd A\Rightarrow /A\Rightarrow;A\Rightarrow;\Rightarrow B;B\Rightarrow}\,[\rhd\Rightarrow]}{A,\rhd(A\to B),\rhd A\Rightarrow /A\Rightarrow;\Rightarrow B;B\Rightarrow}\,[\widetilde{\mathbf{T}}]}{A,A,\rhd(A\to B),\rhd A\Rightarrow /\Rightarrow B;B\Rightarrow}\,[\widetilde{\mathbf{T}}]}{A,\rhd(A\to B),\rhd A\Rightarrow /\Rightarrow B;B\Rightarrow}\,[C\Rightarrow]}{A,\rhd(A\to B),\rhd A\Rightarrow\rhd B/B\Rightarrow}\,[\Rightarrow\rhd_R]}{A,\rhd(A\to B),\rhd A\Rightarrow\rhd B,\rhd B}\,[\Rightarrow\rhd_L]}{A,\rhd(A\to B),\rhd A\Rightarrow\rhd B}\,[\Rightarrow C]}{\Rightarrow A\to\big(\rhd(A\to B)\to(\rhd A\to\rhd B)\big)}\,[\Rightarrow\to],3x$$

An example of a proof in $\mathbb{N}\$\mathbf{B}^{\rhd}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow /\rhd A\Rightarrow /A\Rightarrow A\qquad \Rightarrow /\rhd A\Rightarrow /A\Rightarrow A}{\Rightarrow /\rhd A\Rightarrow /\Rightarrow A;A\Rightarrow}\,[\rhd\Rightarrow]}{A\Rightarrow /\rhd A\Rightarrow /\Rightarrow A}\,[\widetilde{\mathbf{B}}]}{A\Rightarrow /\rhd A\Rightarrow A}\,[\widetilde{\mathbf{T}}]}{A\Rightarrow /\Rightarrow\rhd A\to A}\,[\Rightarrow\to]}{A\Rightarrow\rhd(\rhd A\to A)}\,[\Rightarrow\rhd_R]}{\Rightarrow A\to\rhd(\rhd A\to A)}\,[\Rightarrow\to]$$

An example of a proof in $\mathbb{N}\$\mathbf{K4}^{\rhd}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\rhd A \Rightarrow /A \Rightarrow A \qquad \rhd A \Rightarrow /A \Rightarrow A}{\rhd A \Rightarrow / \Rightarrow A; A \Rightarrow}\,[\rhd \Rightarrow]}{\rhd A \Rightarrow / \Rightarrow / \Rightarrow A; A \Rightarrow}\,[\widetilde{\mathbf{4}}]}{\rhd A \Rightarrow / \Rightarrow \rhd A/A \Rightarrow}\,[\Rightarrow \rhd_R]}{\rhd A \Rightarrow / \Rightarrow \rhd A, \rhd A}\,[\Rightarrow \rhd_L]}{\rhd A \Rightarrow / \Rightarrow \rhd A}\,[\Rightarrow C]}{\rhd A \Rightarrow \rhd\rhd A}\,[\Rightarrow \rhd_R]}{\Rightarrow \rhd A \rightarrow \rhd\rhd A}\,[\Rightarrow\rightarrow]$$

2.4.1 Soundness and completeness

Following Poggiolesi [157], let us introduce the following definition and lemma.

Definition 23. [157, Definition 8.1] Let $\mathcal{M} = \langle W, R, \vartheta \rangle$, $w \in W$, and $\mathfrak{N}$ be a nested sequent. Then $w \models_{\mathcal{M}} \mathfrak{N}$ is inductively defined as follows:

- $w \models_{\mathcal{M}} \Gamma \Rightarrow \Delta$ iff $\exists B \in \Gamma (w \not\models_{\mathcal{M}} B)$ or $\exists C \in \Delta (w \models_{\mathcal{M}} C)$,
- $w \models_{\mathcal{M}} \Gamma \Rightarrow \Delta/X$ iff $w \models_{\mathcal{M}} \Gamma \Rightarrow \Delta$ or $\exists \mathfrak{N} \in X\ \forall_{u \in W}(R(w, u)$ implies $u \models_{\mathcal{M}} \mathfrak{N})$.

For any multiset of formulas Γ, $w \models_{\mathcal{M}} \Gamma$ is defined as $\exists_{A \in \Gamma}(w \models_{\mathcal{M}} A)$.

We write $w \models^*_{\mathcal{M}} \mathfrak{X}$ iff $\forall_{u \in W}(R(w, u)$ implies $u \models_{\mathcal{M}} \mathfrak{X})$, where $\mathfrak{X}$ is a metavariable for a formula, a multiset of sequents, a nested sequent, and a multiset of a nested sequent.

Hence, we can write $w \models_{\mathcal{M}} \Gamma \Rightarrow \Delta/X$ iff $w \models_{\mathcal{M}} \Gamma \Rightarrow \Delta$ or $\exists \mathfrak{N} \in X$, $w \models^*_{\mathcal{M}} \mathfrak{N}$.

Following [157], we will write $w \models_{\mathfrak{C}f} \mathfrak{X}$ for the fact that $w \models_{\mathcal{M}} \mathfrak{X}$, for any nested sequent or sequent $\mathfrak{X}$, any class of frames $\mathfrak{C}$ with the property f, and any model $\mathcal{M}$ based on any frame that belongs to $\mathfrak{C}f$.

Lemma 24. *[157, Lemma 8.2]*

- *For any sequents $\Gamma \Rightarrow \Delta$ and $\Theta \Rightarrow \Lambda$ as well as any nested sequent $\mathfrak{N}$,*

$$\textit{if} \quad \forall w (w \models_{\mathfrak{C}f} \Gamma \Rightarrow \Delta \textit{ implies } \models_{\mathfrak{C}f} \Theta \Rightarrow \Lambda), \textit{ then}$$

$$\forall w (w \models_{\mathfrak{C}f} \mathfrak{N}[\Gamma \Rightarrow \Delta] \textit{ implies } \models_{\mathfrak{C}f} \mathfrak{N}[\Gamma \Rightarrow \Delta/\Theta \Rightarrow \Lambda]).$$

- *For any nested sequents $\mathfrak{K}$, $\mathfrak{L}$, and $\mathfrak{N}$,*

$$\textit{if} \quad \forall w (w \models_{\mathfrak{C}f} \mathfrak{K} \textit{ implies } \models_{\mathfrak{C}f} \mathfrak{L}), \textit{ then}$$

$$\forall w (w \models_{\mathfrak{C}f} \mathfrak{N}[\mathfrak{K}] \textit{ implies } \models_{\mathfrak{C}f} \mathfrak{N}[\mathfrak{K}/\mathfrak{L}]).$$

Lemma 25. *Let $\clubsuit \in \{\rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$. All the rules of $\mathbb{N}\$\mathbf{K}^{\clubsuit}$ are sound.*

Proof. Similarly to [157, Theorem 8.3]. As an example, we show the rule $[\Rightarrow \rhd_L]$. Suppose that $\mathbf{K}^{\clubsuit} \models \Gamma \Rightarrow \Delta/A \Rightarrow$. Then $\forall w (w \models_{\mathfrak{C}f} \Gamma \Rightarrow \Delta$ or $\not\models^*_{\mathfrak{C}f} A)$. Then $\forall w (w \models_{\mathfrak{C}f} \Gamma \Rightarrow \Delta$ or $\not\models_{\mathfrak{C}f} \rhd A)$. Thus, by Lemma 24, $\mathbf{K}^{\clubsuit} \models \Gamma \Rightarrow \Delta, \rhd A$. □

Theorem 26. *For any nested sequent $\mathfrak{N}$, if $\mathbb{N}\$\mathbf{K}^{\clubsuit} \vdash \mathfrak{N}$, then $\mathbf{K}^{\clubsuit} \models \mathfrak{N}$.*

Proof. By induction on the height of the derivation, using Lemma 25. □

Theorem 27. *Let $\clubsuit \in \{\rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$ and $\mathbf{X}_1, \ldots \mathbf{X}_m \in \{\mathbf{T}, \mathbf{D}, \mathbf{4}, \mathbf{B}\}$. For any nested sequent $\mathfrak{N}$, if $\mathbb{N}\$\mathbf{KX}_1, \ldots \mathbf{X}^{\clubsuit}_m \vdash \mathfrak{N}$, then $\mathbf{KX}_1, \ldots \mathbf{X}^{\clubsuit}_m \models \mathfrak{N}$.*

Proof. Follows from Theorem 26 and soundness of the special structural rules established in [157]. □

We present a semantic completeness proof for a nested sequent calculus for $\mathbf{K}^{\clubsuit}$, where $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$, following Poggiolesi [157] (she herself follows Brünnler [22]). Completeness for the extensions of $\mathbf{K}^{\clubsuit}$ follows from the established in [157] correspondence of the structural rules $[\widetilde{\mathbf{D}}]$, $[\widetilde{\mathbf{T}}]$, $[\widetilde{\mathbf{4}}]$, $[\widetilde{\mathbf{B}}]$ for the properties of the accessibility relation. The calculus in question should be slightly reformulated (the result of the reformulation will be called $\mathbb{N}\$\mathbf{K}^{\clubsuit+}$). For each rule $\mathfrak{R}$, we define a rule $\mathfrak{R}^+$ which has the principal formula from the conclusion in its premises. At that we have $\mathfrak{R} = \mathfrak{R}^+$ for the following rules: $[\triangleright \Rightarrow]$, $[\Rightarrow \blacktriangleright]$, $[\circ \Rightarrow]$, $[\Rightarrow \bullet]$, $[\widetilde{\circ} \Rightarrow]$, $[\Rightarrow \widetilde{\bullet}]$, $[\Rightarrow \sim]$, $[\dot{\sim} \Rightarrow]$. For the other rules $\mathfrak{R}^+$ is as follows:

$$[\neg \Rightarrow]^+ \ \frac{\mathfrak{N}[\neg A, \Gamma \Rightarrow \Delta, A]}{\mathfrak{N}[\neg A, \Gamma \Rightarrow \Delta]} \qquad [\Rightarrow \neg]^+ \ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta, \neg A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg A]}$$

$$[\wedge \Rightarrow]^+ \ \frac{\mathfrak{N}[A, B, A \wedge B, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[A \wedge B, \Gamma \Rightarrow \Delta]} \qquad [\Rightarrow \wedge]^+ \ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A, A \wedge B] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta, B, A \wedge B]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A \wedge B]}$$

$$[\vee \Rightarrow]^+ \ \frac{\mathfrak{N}[A \vee B, A, \Gamma \Rightarrow \Delta] \quad \mathfrak{N}[A \vee B, B, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[A \vee B, \Gamma \Rightarrow \Delta]} \qquad [\Rightarrow \vee]^+ \ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A, B, A \vee B]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A \vee B]}$$

$$[\to\Rightarrow]^+ \ \frac{\mathfrak{N}[A \to B, \Gamma \Rightarrow \Delta, A] \quad \mathfrak{N}[A \to B, B, \Gamma \Rightarrow \Delta]}{\mathfrak{N}[A \to B, \Gamma \Rightarrow \Delta]} \qquad [\Rightarrow\to]^+ \ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta, B, A \to B]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A \to B]}$$

$$[\leftrightarrow\Rightarrow]^+ \ \frac{\mathfrak{N}[B, \Gamma \Rightarrow \Delta, A, A \leftrightarrow B] \quad \mathfrak{N}[A, \Gamma \Rightarrow \Delta, B, A \leftrightarrow B]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A \leftrightarrow B]}$$

$$[\Rightarrow\leftrightarrow]^+ \ \frac{\mathfrak{N}[A \leftrightarrow B, A, B, \Gamma \Rightarrow \Delta] \quad \mathfrak{N}[A \leftrightarrow B, \Gamma \Rightarrow \Delta, A, B]}{\mathfrak{N}[A \leftrightarrow B, \Gamma \Rightarrow \Delta]}$$

$$[\Rightarrow \triangleright_L]^+ \ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \triangleright A / A \Rightarrow]^{\dagger}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \triangleright A]} \qquad [\Rightarrow \triangleright_R]^+ \ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \triangleright A / \Rightarrow A]^{\ddagger}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \triangleright A]}$$

† where the sequent $\Gamma \Rightarrow \Delta, \triangleright A$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the right side.
‡ where the sequent $\Gamma \Rightarrow \Delta, \triangleright A$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the left side.

$$[\blacktriangleright \Rightarrow_L]^+ \ \frac{\mathfrak{N}[\blacktriangleright A, \Gamma \Rightarrow \Delta / A \Rightarrow]^{\dagger}}{\mathfrak{N}[\blacktriangleright A, \Gamma \Rightarrow \Delta]} \qquad [\blacktriangleright \Rightarrow_R]^+ \ \frac{\mathfrak{N}[\blacktriangleright A, \Gamma \Rightarrow \Delta / \Rightarrow A]^{\ddagger}}{\mathfrak{N}[\blacktriangleright A, \Gamma \Rightarrow \Delta]}$$

† where the sequent $\blacktriangleright A, \Gamma \Rightarrow \Delta$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the right side.
‡ where the sequent $\blacktriangleright A, \Gamma \Rightarrow \Delta$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the left side.

$$[\Rightarrow \circ_L]^+ \ \frac{\mathfrak{N}[A, \Gamma \Rightarrow \Delta, \circ A]^{\dagger}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \circ A]} \qquad [\Rightarrow \circ_R]^+ \ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \circ A / \Rightarrow A]^{\ddagger}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \circ A]}$$

† where the sequent $A, \Gamma \Rightarrow \Delta, \circ A$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the right side.
‡ where the sequent $\Gamma \Rightarrow \Delta, \circ A$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the left side.

$$[\bullet \Rightarrow_L]^+ \ \dfrac{\mathfrak{N}[\bullet A, A, \Gamma \Rightarrow \Delta]^{\dagger}}{\mathfrak{N}[\bullet A, \Gamma \Rightarrow \Delta]} \qquad [\bullet \Rightarrow_R]^+ \ \dfrac{\mathfrak{N}[\bullet A, \Gamma \Rightarrow \Delta / \Rightarrow A]^{\ddagger}}{\mathfrak{N}[\bullet A, \Gamma \Rightarrow \Delta]}$$

† where the sequent $\bullet A, A, \Gamma \Rightarrow \Delta$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the right side.
‡ where the sequent $\bullet A, \Gamma \Rightarrow \Delta$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the left side.

$$[\Rightarrow \widetilde{\circ}_L]^+ \ \dfrac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \widetilde{\circ} A / A \Rightarrow]^{\dagger}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \widetilde{\circ} A]} \qquad [\Rightarrow \widetilde{\circ}_R]^+ \ \dfrac{\mathfrak{N}[\Gamma \Rightarrow \Delta, A, \widetilde{\circ} A]^{\ddagger}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \widetilde{\circ} A]}$$

† where the sequent $\Gamma \Rightarrow \Delta, \widetilde{\circ} A$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the right side.
‡ where the sequent $\Gamma \Rightarrow \Delta, A, \widetilde{\circ} A$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the left side.

$$[\widetilde{\bullet} \Rightarrow_L]^+ \ \dfrac{\mathfrak{N}[\widetilde{\bullet} A, \Gamma \Rightarrow \Delta / A \Rightarrow]^{\dagger}}{\mathfrak{N}[\widetilde{\bullet} A, \Gamma \Rightarrow \Delta]} \qquad [\widetilde{\bullet} \Rightarrow_R]^+ \ \dfrac{\mathfrak{N}[\widetilde{\bullet} A, \Gamma \Rightarrow \Delta, A]^{\ddagger}}{\mathfrak{N}[\widetilde{\bullet} A, \Gamma \Rightarrow \Delta]}$$

† where the sequent $\widetilde{\bullet} A, \Gamma \Rightarrow \Delta$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the right side.
‡ where the sequent $\widetilde{\bullet} A, \Gamma \Rightarrow \Delta, A$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the left side.

$$[\sim \Rightarrow]^+ \ \dfrac{\mathfrak{N}[\sim A, \Gamma \Rightarrow \Delta / \Rightarrow A]^{\ddagger}}{\mathfrak{N}[\sim A, \Gamma \Rightarrow \Delta]} \qquad [\Rightarrow \dot{\sim}]^+ \ \dfrac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \dot{\sim} A / A \Rightarrow]^{\dagger}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \dot{\sim} A]}$$

† where the sequent $\Gamma \Rightarrow \Delta, \dot{\sim} A$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the right side.
‡ where the sequent $\sim A, \Gamma \Rightarrow \Delta$ does not have any immediate successive sequent (or more succinctly, childsequent) that contains the formula A on the left side.

Definition 28. [157, Definition 8.4] The *set nested sequent* of a nested sequent $\Gamma \Rightarrow \Delta / \mathfrak{M}_1; \ldots; \mathfrak{M}_m$ is the underlying set of $\Theta \Rightarrow \Lambda / \mathfrak{N}_1; \ldots; \mathfrak{N}_m$, where $\mathfrak{N}_1; \ldots; \mathfrak{N}_m$ are the set nested sequent of $\mathfrak{M}_1; \ldots; \mathfrak{M}_m$. Clearly, the set nested sequent of a nested sequent is still a nested sequent since a set is a multiset.

For any rule $\mathfrak{R}^+$ we have the proviso that for all of its premises, the set nested sequent is different from the set nested sequent of the conclusion.

Lemma 29. *Let* $\clubsuit \in \{\triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$. *For any nested sequent* $\mathfrak{N}$*, it holds that if* $\mathbb{N}\mathbb{S}\mathbf{K}^{\clubsuit +} \vdash \mathfrak{N}$*, then* $\mathbb{N}\mathbb{S}\mathbf{K}^{\clubsuit} \vdash \mathfrak{N}$.

Proof. By induction on the height of derivations in $\mathbb{N}\mathbb{S}\mathbf{K}^{\clubsuit +}$, using contraction and weakening. □

Definition 30. [157, Definition 8.10] A leaf of a nested sequent is *cyclic* if in its branch there exists a sequent that contains the same set of formulas.

Definition 31. [157, Definition 8.11] A sequent of a nested sequent is *finished* for a nested sequent calculus $\mathbb{N}\mathbb{S}\mathbf{K}^{\clubsuit +}$ if no rule of that calculus applies to one of its formulas. A nested sequent is finished for a nested sequent calculus $\mathbb{N}\mathbb{S}\mathbf{K}^{\clubsuit +}$ if all sequents that compose it are finished or cyclic.

Definition 32. [157, Definition 8.12] We define a procedure *prove*$(\mathfrak{N}, \mathbb{NS}\mathbf{K}^{\clubsuit+})$, which takes a nested sequent $\mathfrak{N}$ and a calculus $\mathbb{NS}\mathbf{K}^{\clubsuit+}$, and builds a derivation tree for $\mathfrak{N}$ by applying rules from that calculus to non-initial and unfinished derivation leaves in the bottom-up fashion, as follows:

1. keep applying all the rules of $\mathbb{NS}\mathbf{K}^{\clubsuit+}$ which are not the rules with the provisos above indicated as † and ‡ long as possible;
2. wherever possible, apply these rules with the † and ‡ provisos once.

Repeat this operation until each non-initial derivation leaf of the tree-hypersequent $\mathfrak{N}$ is finished. If *prove*$(\mathfrak{N}, \mathbb{NS}\mathbf{K}^{\clubsuit+})$ terminates and all derivation leaves are initial, then it succeeds; otherwise, i.e., if it terminates and there is a non-initial derivation leaf, it fails.

Definition 33. [157, Definition 8.13] The *size* of a nested sequent $\mathfrak{N}$, $s(\mathfrak{N})$, is the number of sequents that compose it. The *set of subformulas* of a nested sequent $\mathfrak{N}$, a nested sequent $sf(\mathfrak{N})$, is the set of all subformulas of all formulas that compose all sequents that belong to the nested sequent.

Definition 34. [157, Definition 8.15] A nested sequent $\mathfrak{N}$ is an *immediate subtree* of a nested sequent $\mathfrak{M}$ if $\mathfrak{M}$ is of the form $\Gamma \Rightarrow \Delta/\mathfrak{N}; \mathfrak{N}_1; ...; \mathfrak{N}_n$. It is a proper subtree if it is an immediate subtree either of $\mathfrak{M}$ or of a proper subtree of $\mathfrak{M}$, and it is a subtree if it is either a proper subtree of $\mathfrak{M}$ or $\mathfrak{M} = \mathfrak{N}$. The set of all subtrees of $\mathfrak{M}$ is denoted by $st(\mathfrak{M})$.

Lemma 35. *[157, Lemma 8.14] For any nested sequent $\mathfrak{N}$, the procedure prove$(\mathfrak{N}, \mathbb{NS}\mathbf{K}^{\clubsuit+})$ terminates after at most $2^{|sf(\mathfrak{N})|}$ iterations.*

Theorem 36. *Let $\clubsuit \in \{\rhd, \blacktriangleright, \circ, \bullet, \tilde{\circ}, \tilde{\bullet}, \sim, \dot{\sim}\}$. For any nested sequent $\mathfrak{N}$, it holds that*

(1) *if $\mathbf{K}^{\clubsuit} \models \mathfrak{N}$, then $\mathbb{NS}\mathbf{K}^{\clubsuit} \vdash \mathfrak{N}$,*

(2) *if prove$(\mathfrak{N}, \mathbb{NS}\mathbf{K}^{\clubsuit+})$ fails, then $\mathbf{K}^{\clubsuit+} \not\models \mathfrak{N}$.*

Proof. Similarly to [157, Theorem 8.16]. The contraposition of (1) follows from (2). Suppose that $\mathbb{NS}\mathbf{K}^{\clubsuit} \not\vdash \mathfrak{N}$. By Lemma 29, $\mathbb{NS}\mathbf{K}^{\clubsuit+} \not\vdash \mathfrak{N}$. By Lemma 35, *prove*$(\mathfrak{N}, \mathbb{NS}\mathbf{K}^{\clubsuit+})$ has to fail. Let us define a countermodel for $\mathfrak{N}$.

Let $\mathfrak{N}^*$ be the set nested sequent obtained from a nested sequent which is not an axiom. Let $\mathbb{Y}$ be the set of all cyclic leaves in $\mathfrak{N}^*$. Let $W = st(\mathfrak{N}^*) \setminus \mathbb{Y}$. Let $f : \mathbb{Y} \mapsto W$ be some function which maps a cyclic leaf to a nested sequent in W whose root carries the same set of formulas, and extend f to $st(\mathfrak{N}^*)$ by the identity on W. Define a binary relation R on W such that $R(\mathfrak{K}, \mathfrak{L})$ if, and only if, either (i) $\mathfrak{L}$ is an immediate subtree of $\mathfrak{K}$, or (ii) $\mathfrak{K}$ has an immediate subtree $\mathfrak{M} \in \mathbb{Y}$ and $f(\mathfrak{M}) = \mathfrak{L}$. Let $\vartheta(\mathfrak{N}, p)$ such that $\vartheta(\mathfrak{N}, p) = 1$, if p occurs on the left side of a sequent $\Gamma \Rightarrow \Delta \in \mathfrak{N}$, and $\vartheta(\mathfrak{N}, p) = 0$ otherwise.

Let $\mathcal{M} = (W, R, \vartheta)$. As an example, we consider the case $\clubsuit = \rhd$.

Claim 1. For all $\mathfrak{K}, \mathfrak{L} \in W$ such that $R(\mathfrak{K}, \mathfrak{L})$, for all A occurring on the left side of a sequent that belongs to the nested sequent $\mathfrak{N}$, we have: if $\rhd A \in \mathfrak{K}$, then $A \in \mathfrak{L}$ or $\neg A \in \mathfrak{L}$. By the definition of R and the rules $(\rhd \Rightarrow)$ and $(\neg \Rightarrow)$, we have A in (the root sequent of) all immediate subtrees of $\mathfrak{K}$.

Claim 2. For all $\mathfrak{K} \in W$, we have:

- for all $A \in \mathfrak{K}$ such that they occur on the left side of the sequent, $\mathfrak{K} \models_{\mathcal{M}} A$,
- for all $A \in \mathfrak{K}$ such that they occur on the right side of the sequent, $\mathfrak{K} \not\models_{\mathcal{M}} A$.

By induction on the complexity of the formula A. The basic case follows from the definition of the valuation. The propositional cases are rather obvious. Let $A = \rhd B$. Suppose that it occurs on the right side of the sequent, then by the rules $(\Rightarrow \rhd_L)$ and $(\Rightarrow \rhd_R)$ as well as the rules for negations, we have at least one $\mathfrak{M} \in \mathfrak{K}$ with $B \in \mathfrak{M}$ and at least one $\mathfrak{M}' \in \mathfrak{K}$ with $\neg B \in \mathfrak{M}'$. By the inductive hypothesis, $\mathfrak{M} \not\models_{\mathcal{M}} B$ and $\mathfrak{M}' \not\models_{\mathcal{M}} \neg B$ (that is $\mathfrak{M}' \models_{\mathcal{M}} B$). Thus, $\mathfrak{K} \not\models_{\mathcal{M}} \rhd B$. Suppose that $\rhd B$ occurs on the left side of the sequent. By Claim 1, $B \in \mathfrak{M}$, for all $\mathfrak{M}$ such that $R(\mathfrak{K}, \mathfrak{M})$, or $\neg B \in \mathfrak{M}'$, for all $\mathfrak{M}'$ such that $R(\mathfrak{K}, \mathfrak{M}')$. Using the inductive hypothesis, $\mathfrak{K} \models_{\mathcal{M}} \rhd B$.

Claim 3. For all $\mathfrak{K} \in st(\mathfrak{N}^*)$, $f(\mathfrak{K}) \not\models_{\mathcal{M}} \mathfrak{K}$.

By induction on the complexity of the nested sequent $\mathfrak{K}$, using Claim 2. See [157, Theorem 8.16].

Since all rules seen top-down preserve countermodels, Claim 3 implies that $\not\models_{\mathcal{M}} \mathfrak{N}$. □

Theorem 37 (Cut admissibility)**.** *Let* $\clubsuit \in \{\rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$, $\mathbf{L} = \mathbb{N}\$\mathbf{K}^{\clubsuit}$*, and* $\mathfrak{N}$ *be a nested sequent. Then* $\vdash_{\mathbb{N}\$\mathbf{L}} \mathfrak{N}$ *implies that there is a cut-free proof of* $\mathfrak{N}$ *in* $\mathbb{N}\$\mathbf{L}$*.*

Proof. Follows from Theorem 36 and the fact that in its proof, the rule of cut has not been used. □

Theorem 38. *Let* $\clubsuit \in \{\rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$ *and* $\mathbf{X}_1, \ldots \mathbf{X}_m \in \{\mathbf{T}, \mathbf{D}, \mathbf{4}, \mathbf{B}\}$*. For any nested sequent* $\mathfrak{N}$*, if* $\mathbf{KX}_1, \ldots \mathbf{X}_m^{\clubsuit} \models \mathfrak{N}$*, then* $\mathbb{N}\$\mathbf{KX}_1, \ldots \mathbf{X}_m^{\clubsuit} \vdash \mathfrak{N}$*.*

Proof. Follows from Theorem 36 and the fact (which follows from the results of [157]) that special structural rules correspond to the properties of the accessibility relation. □

2.4.2 Constructive proof of the cut admissibility theorem

Let us prove a constructive cut admissibility for the calculi in question, following the methods from [157], where such proof was given for the logics built in the language $\mathscr{L}_{\Box}$.

Lemma 39. *Given three zoom nested sequents* $\mathfrak{K}[*]$, $\mathfrak{L}[*]$*, and* $\mathfrak{M}[*]$ *such that* $\mathfrak{K}[*] \approx \mathfrak{L}[*] \approx \mathfrak{M}[*]$*, if there is a rule* $\mathfrak{R}$ *of* $\mathbb{N}\$\mathbf{L}$ *and a sequent* Γ *such that*

$$\mathfrak{R}\ \frac{\mathfrak{L}[\Gamma]}{\mathfrak{K}[\Gamma]}$$

then, for any Δ*, we have that*

$$\mathfrak{R}\ \frac{\mathfrak{L} \otimes \mathfrak{M}[\Delta]}{\mathfrak{K} \otimes \mathfrak{M}[\Delta]}$$

Proof. We follow the method from [157, p. 143, Lemma 7.1]. By induction on the form of nested sequents $\mathfrak{K}[*]$, $\mathfrak{L}[*]$, and $\mathfrak{M}[*]$. The proof consists of the following parts:

(A) $\mathfrak{K}[*]$, $\mathfrak{L}[*]$, and $\mathfrak{M}[*] \equiv *$.

(B) $\mathfrak{K}[*] \equiv */X$, $\mathfrak{L}[*] \equiv */Y$, and $\mathfrak{M}[*] \equiv */Z$.

(C) $\mathfrak{K}[*] \equiv \Gamma_1 \Rightarrow \Delta_1/\mathfrak{K}'[*]; X$, $\mathfrak{L}[*] \equiv \Gamma_2 \Rightarrow \Delta_2/\mathfrak{L}'[*]; Y$, and $\mathfrak{M}[*] \equiv \Gamma_3 \Rightarrow \Delta_3/\mathfrak{M}'[*]; Z$.

Let

$$\mathfrak{R}\ \frac{\Gamma_1 \Rightarrow \Delta_1/\mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X}{\Gamma_2 \Rightarrow \Delta_2/\mathfrak{L}'[\Theta_2 \Rightarrow \Lambda_2]; Y}$$

For any $\Pi \Rightarrow \Sigma$, we have that

$$\mathfrak{R}\ \frac{\Gamma_1, \Gamma_3 \Rightarrow \Delta_1, \Delta_3/\mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1] \otimes \mathfrak{M}'[\Pi \Rightarrow \Sigma]; X; Z}{\Gamma_2, \Gamma_3 \Rightarrow \Delta_2, \Delta_3/\mathfrak{L}'[\Theta_2 \Rightarrow \Lambda_2] \otimes \mathfrak{M}'[\Pi \Rightarrow \Sigma]; Y; Z}$$

The following subcases are distinguished.

(C.1) The rule $\mathfrak{R}$ operates on $\Gamma_1 \Rightarrow \Delta_1$:

(C.1.1) The rule $\mathfrak{R}$ operates on $\Gamma_1 \Rightarrow \Delta_1$ only,

(C.1.2) The rule $\mathfrak{R}$ operates between $\Gamma_1 \Rightarrow \Delta_1$ and $\mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X$.

(C.2) The rule $\mathfrak{R}$ operates on X.

(C.3) The rule $\mathfrak{R}$ operates on $\mathfrak{K}'$.

The cases (A), (B), (C.1.1), (C.2), and (C.3) are proven in [157]. The case (C.1.2) deals with modal rules. Let us consider it on the example of the rules for $\rhd$.

Let $\mathfrak{R}$ be the rule $[\Rightarrow \rhd_L]$. There are two subcases: (i) $\mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X$ is of the form $B \Rightarrow ; \mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X'$, (ii) $\mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X$ is of the form $B \Rightarrow ; \mathfrak{K}''[\Theta_1 \Rightarrow \Lambda_1]; X$. As an example, we show the case (i),

$$\frac{\Gamma_1 \Rightarrow \Delta_1 / B \Rightarrow ; \mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X'}{\Gamma_1 \Rightarrow \Delta_1, \rhd B \Rightarrow ; \mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1]; X'}$$

For any $\Pi \Rightarrow \Sigma$, we have that

$$\frac{\Gamma_1, \Gamma_3 \Rightarrow \Delta_1, \Delta_3 / B \Rightarrow ; \mathfrak{K}'[\Theta_1 \Rightarrow \Lambda_1] \otimes \mathfrak{M}'[\Pi \Rightarrow \Sigma]; X'; Z}{\Gamma_1, \Gamma_3 \Rightarrow \Delta_1, \Delta_3, \rhd B / \mathfrak{L}'[\Theta_2 \Rightarrow \Lambda_2] \otimes \mathfrak{M}'[\Pi \Rightarrow \Sigma]; X'; Z}$$

The cases of the rules $[\Rightarrow \rhd_R]$ and $[\rhd \Rightarrow]$ are treated similarly. □

Lemma 40. *Let $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ and $\mathfrak{M}[A, \Pi \Rightarrow \Sigma]$ be such that $\mathfrak{N}[\Gamma \Rightarrow \Delta, A] \approx \mathfrak{M}[A, \Pi \Rightarrow \Sigma]$. If*

$$[\mathrm{Cut}]\ \frac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2\\ \mathfrak{N}[\Gamma \Rightarrow \Delta, A] & \mathfrak{M}[A, \Pi \Rightarrow \Sigma]\end{array}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma]}$$

and $\mathfrak{D}_1$ and $\mathfrak{D}_2$ do not contain any other application of the cut-rule, then we can construct a derivation of $\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma]$ with no application of the cut-rule.

Proof. Similarly to the proof of Lemma 7.2 from [157]. By a double induction on the complexity of the cut-formula $\mathfrak{c}(A)$[13] and on the sum of the heights of the derivations of the premises of the cut-rule. The cases are distinguished according to the last rule applied to the left premise.

Case 1. $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ is an axiom. This case is considered in the proof of Lemma 7.2 from [157]: either the conclusion is also an axiom or it can be inferred from $\mathfrak{M}[A, \Pi \Rightarrow \Sigma]$ by internal and external weakening rules.

Case 2. $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ is derived by a rule $\mathfrak{R}$ such that A is not principal. The case is solved by induction on the sum of the heights of the derivations of the premises of the cut-rule, using Lemma 39.

Subcase 2.1. $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ was obtained by the rule $[\Rightarrow \rhd_L]$. The following application of cut

$$\frac{\dfrac{\begin{array}{c}\mathfrak{D}_1'\\ \mathfrak{N}[\Gamma \Rightarrow \Delta, A / B \Rightarrow]\end{array}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, A, \rhd B]}[\Rightarrow \rhd_L] \qquad \begin{array}{c}\mathfrak{D}_2\\ \mathfrak{M}[A, \Pi \Rightarrow \Sigma]\end{array}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma, \rhd B]}[\mathrm{Cut}]$$

is reduced to the subsequent deduction, where cut is applied to a formula of a lower height:

$$\dfrac{\dfrac{\begin{array}{cc}\mathfrak{D}_1' & \mathfrak{D}_2\\ \mathfrak{N}[\Gamma \Rightarrow \Delta, A / B \Rightarrow] & \mathfrak{M}[A, \Pi \Rightarrow \Sigma]\end{array}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma / B \Rightarrow]}[\mathrm{Cut}]}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma, \rhd B]}[\Rightarrow \rhd_L]$$

[13]The definition is standard: $\mathfrak{c}(p) = 1$, $\mathfrak{c}(*A) = \mathfrak{c}(A)+1$, where $*$ is an unary connective, $\mathfrak{c}(A \star B) = \max(\mathfrak{c}(A), \mathfrak{c}(B))+1$, where $*$ is a binary connective.

Subcase 2.2. $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ was obtained by the rule $[\Rightarrow \rhd_R]$. Similarly to the previous subcase.

Subcase 2.3. $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ was obtained by the rule $[\rhd \Rightarrow]$. The following application of cut

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1' \\ \mathfrak{N}[\rhd B, \Gamma \Rightarrow \Delta, A/(B, \Theta \Rightarrow \Lambda/X)]\end{matrix} \quad \begin{matrix}\mathfrak{D}_1'' \\ \mathfrak{N}[\rhd B, \Gamma \Rightarrow \Delta, A/(\Xi \Rightarrow \Upsilon, B/Y)]\end{matrix}}{\mathfrak{N}[\rhd B, \Gamma \Rightarrow \Delta, A/(\Theta \Rightarrow \Lambda/X); (\Xi \Rightarrow \Upsilon/Y)]}[\rhd \Rightarrow] \quad \begin{matrix}\mathfrak{D}_2 \\ \mathfrak{M}[A, \Pi \Rightarrow \Sigma]\end{matrix}}{\mathfrak{N} \otimes \mathfrak{M}[\rhd B, \Gamma, \Pi \Rightarrow \Delta, \Sigma/(\Theta \Rightarrow \Lambda/X); (\Xi \Rightarrow \Upsilon/Y)]}$$

is reduced to the subsequent deduction, where cut is applied to a formula of a lower height:

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1' \\ \mathfrak{N}[\rhd B, \Gamma \Rightarrow \Delta, A/(B, \Theta \Rightarrow \Lambda/X)]\end{matrix} \quad \begin{matrix}\mathfrak{D}_2 \\ \mathfrak{M}[A, \Pi \Rightarrow \Sigma]\end{matrix}}{\mathfrak{N} \otimes \mathfrak{M}[\rhd B, \Gamma, \Pi \Rightarrow \Delta, \Sigma/(B, \Theta \Rightarrow \Lambda/X)]} \quad \dfrac{\begin{matrix}\mathfrak{D}_1'' \\ \mathfrak{N}[\rhd B, \Gamma \Rightarrow \Delta, A/(\Xi \Rightarrow \Upsilon, B/Y)]\end{matrix} \quad \begin{matrix}\mathfrak{D}_2 \\ \mathfrak{M}[A, \Pi \Rightarrow \Sigma]\end{matrix}}{\mathfrak{N} \otimes \mathfrak{M}[\rhd B, \Gamma, \Pi \Rightarrow \Delta, \Sigma/(\Xi \Rightarrow \Upsilon, B/Y)]}}{\mathfrak{N} \otimes \mathfrak{M}[\rhd B, \Gamma, \Pi \Rightarrow \Delta, \Sigma/(\Theta \Rightarrow \Lambda/X); (\Xi \Rightarrow \Upsilon/Y)]}$$

The subcases produced by other connectives are treated similarly.

Case 3. $\mathfrak{N}[\Gamma \Rightarrow \Delta, A]$ is derived by a rule $\mathfrak{R}$ such that A is principal. The cases where $\mathfrak{R}$ is a propositional rule (or a rule for $\Box$ or $\Diamond$) are covered in [157]. As an example, we consider the case when $\mathfrak{R}$ is a rule for $\rhd$, $[\Rightarrow \rhd_L]$.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1' \\ \mathfrak{N}[\Gamma \Rightarrow \Delta/B \Rightarrow]\end{matrix}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \rhd B]}[\Rightarrow \rhd_L] \quad \begin{matrix}\mathfrak{D}_2 \\ \mathfrak{M}[\rhd B, \Pi \Rightarrow \Sigma]\end{matrix}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma]}[\text{Cut}]$$

We need to consider the ways $\mathfrak{M}[A, \Pi \Rightarrow \Sigma]$ could be derived. If it is an axiom, then we go to the Case 1. If $\rhd B$ is not the principal formula in $\mathfrak{D}_2$, then we go to the Case 2. However, in contrast to the proof from [157], we postulate structural rules $[\widetilde{\mathbf{D}}]$, $[\widetilde{\mathbf{T}}]$, $[\widetilde{\mathbf{4}}]$, $[\widetilde{\mathbf{B}}]$ as primitive ones. In the logics with $\rhd$, the case of the rule $[\widetilde{\mathbf{D}}]$ is not applicable. Let us consider as an example the case of the rule $[\widetilde{\mathbf{T}}]$ (the case of the rule $[\widetilde{\mathbf{B}}]$ is treated similarly, the rule $[\widetilde{\mathbf{4}}]$ due to its shape cannot move $\rhd B$ from one sequent to another). Let $\Pi = \Pi' \cup \Pi''$ and $\Sigma = \Sigma' \cup \Sigma''$.

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1' \\ \mathfrak{N}[\Gamma \Rightarrow \Delta/B \Rightarrow]\end{matrix}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \rhd B]}[\Rightarrow \rhd_L] \quad \dfrac{\begin{matrix}\mathfrak{D}_2' \\ \mathfrak{M}[\Pi' \Rightarrow \Sigma'/\rhd B, \Pi'' \Rightarrow \Sigma'']\end{matrix}}{\mathfrak{M}[\rhd B, \Pi \Rightarrow \Sigma]}[\widetilde{\mathbf{T}}]}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma]}[\text{Cut}]$$

Then the transformation of the deduction is as follows, where cut can be eliminated due to the induction hypothesis regarding the sum of the heights of the derivation of the premises of the rules of cut:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1' \\ \mathfrak{N}[\Gamma \Rightarrow \Delta/B \Rightarrow]\end{matrix}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \rhd B]}[\Rightarrow \rhd_L] \quad \begin{matrix}\mathfrak{D}_2' \\ \mathfrak{M}[\Pi' \Rightarrow \Sigma'/\rhd B, \Pi'' \Rightarrow \Sigma'']\end{matrix}}{\mathfrak{N} \otimes \mathfrak{M}[\Pi' \Rightarrow \Sigma'/\Gamma, \Pi'' \Rightarrow \Sigma'', \Delta]}[\text{Cut}]}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma]}[\widetilde{\mathbf{T}}]$$

The last option is as follows: $\mathfrak{M}[A, \Pi \Rightarrow \Sigma]$ was obtained by $[\rhd \Rightarrow]$. Poggiolesi has more options [157] in a similar situation with $\Box$: $\mathfrak{M}[A, \Pi \Rightarrow \Sigma]$ was obtained by the logical rules $[\mathbf{D}]$, $[\mathbf{T}]$, $[\mathbf{B}]$, $[\mathbf{4}]$, or $[\mathbf{5}]$. These cases constitute the largest part of the rest of her proof and are solved with the help of the corresponding height-preserving admissible structural rules. We follow a different path: we do not have such logical rules, hence such cases do not appear in our proof (instead we have structural rules as primitive ones, but their cases are simple and have been just treated above). So if $\mathfrak{M}[A, \Pi \Rightarrow \Sigma]$ was obtained by $[\rhd \Rightarrow]$, then we have the following deduction:

$$\dfrac{\dfrac{\overset{\mathfrak{D}'_1}{\mathfrak{N}[\Gamma \Rightarrow \Delta / B \Rightarrow]}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \rhd B]}[\Rightarrow \rhd_L] \qquad \dfrac{\overset{\mathfrak{D}'_2}{\mathfrak{N}[\rhd B, \Pi \Rightarrow \Sigma / (B, \Theta \Rightarrow \Lambda / X)]} \qquad \overset{\mathfrak{D}''_2}{\mathfrak{N}[\rhd B, \Pi \Rightarrow \Sigma / (\Xi \Rightarrow \Upsilon, B / Y)]}}{\mathfrak{M}[\rhd B, \Pi \Rightarrow \Sigma / (\Theta \Rightarrow \Lambda / X); (\Xi \Rightarrow \Upsilon / Y)]}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi \Rightarrow \Delta, \Sigma / (\Theta \Rightarrow \Lambda / X); (\Xi \Rightarrow \Upsilon / Y)]}$$

We transform it in the subsequent way. Double lines indicate here several applications of internal weakening rules as well as merge. The first application of [Cut] is eliminable due to the induction hypothesis regarding the sum of the heights of the derivations of the premises, and second application of [Cut] is eliminable due to the induction hypothesis regarding the complexity of the cut-formula.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\mathfrak{D}'_1}{\mathfrak{N}[\Gamma \Rightarrow \Delta / B \Rightarrow]}}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \rhd B]}[\Rightarrow \rhd_L] \qquad \overset{\mathfrak{D}''_2}{\mathfrak{N}[\rhd B, \Pi \Rightarrow \Sigma / (\Xi \Rightarrow \Upsilon, B / Y)]}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi, \Rightarrow \Delta, \Sigma / (\Xi \Rightarrow \Upsilon, B / Y)]}[\text{Cut}] \qquad \overset{\mathfrak{D}'_1}{\mathfrak{N}[\Gamma \Rightarrow \Delta / B \Rightarrow]}}{\mathfrak{N} \otimes \mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Gamma, \Pi, \Rightarrow \Delta, \Delta, \Sigma / (\Xi \Rightarrow \Upsilon / Y)]}[\text{Cut}]}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi, \Rightarrow \Delta, \Sigma / (\Xi \Rightarrow \Upsilon / Y)]}}{\mathfrak{N} \otimes \mathfrak{M}[\Gamma, \Pi, \Rightarrow \Delta, \Sigma / (\Theta \Rightarrow \Lambda / X); (\Xi \Rightarrow \Upsilon / Y)]}[\text{EW}]$$

□

Theorem 41 (Constructive elimination of cut). *Let* $\clubsuit \in \{\rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\widetilde{\sim}}\}$ *and* $\mathbf{X}_1, \ldots \mathbf{X}_m \in \{\mathbf{T}, \mathbf{D}, \mathbf{4}, \mathbf{B}\}$. *Any derivation* $\mathfrak{D}$ *in* $\mathbb{NS}\mathbf{KX}_1, \ldots \mathbf{X}_m^{\clubsuit} \vdash \mathfrak{N}$ *can be effectively transformed into a derivation* $\mathfrak{D}'$, *where there is no application of the rule of cut.*

Proof. By induction on the number of cuts, using Lemma 40. □

2.5 Natural deduction systems for selected logics with non-standard modalities

In the previous sections, we considered a sequent-based approach to modalities: hypersequent calculi for **S5**-style logics and nested sequent calculi for **KX**-style logics, where $\mathbf{X} \subseteq \{\mathbf{T}, \mathbf{D}, \mathbf{4}, \mathbf{B}\}$. In this section, we are going to focus on the natural deduction approach. In the previous sections, the cut admissibility theorem played an important role in our studies: which is not surprising, since it might be called the most important theorem in proof theory. In the case of natural deduction, the normalisation theorem is usually considered as an analogue of cut admissibility. So we are going to deal with the normalisation theorem. At that point, normalisation is always proved for the tree-format natural deduction systems with a Gentzen-Prawitz-style definition of the notion of deduction. This circumstance seriously restricts our possible choice of natural deduction systems for modal logics. We have already mentioned above some of such systems: Prawitz's original approach [161] (later some mistakes in his proof were found by Medeiros and Da Paz [120]) and the approach by Biermann and de Paiva [16]. We would like to focus on the later approach: Biermann and de Paiva have a relatively simple proof of the normalisation theorem, one of their rules can be observed as a general elimination rule. We will show how all their rules can be represented in the form of general elimination or general introduction rules. It is important for us, because of two reasons: we would like to follow Kürbis' strategy in proving the normalisation theorem and such a type of rules is needed for it, and such rules are very helpful in the formulation of the rules for n-ary Boolean connectives by Segerberg's general method [173] (to be considered later in this section) and n-ary three- and four-valued connectives produced by Kooi and Tamminga's correspondence analysis [96, 189, 97] (a form of Segerberg's method). We would like to make this work as uniform as possible: so if Boolean and many-valued cases require general introduction and elimination rules, we would like to have the same type of rules for modalities as well. Kürbis' strategy requires the use of such rules to be applied, and at the same time, it seems to be the most convenient and effective strategy, if one deals with general introduction and elimination rules. Also, such a choice of rules and strategy allows to overcome some difficulties that arose in Prawitz's proof for classical logic [161] (Prawitz considers the language without disjunction, while Kürbis' proof brings it back while used).

Yet another important issue we would like to mention at the beginning of this section: before we prove normalisation for modal logic we need to have a proof of normalisation for classical logic. It sounds obvious, but in contrast to the case of sequent calculus where there is no problem with cut admissibility for classical logic, there are some difficulties with the proof of the normalisation theorem for classical logic. The first proof was presented by Prawitz [161]. However, the choice of language is restricted: no disjunction and no existential quantifier. A brilliant and successful attempt to overcome this difficulty has been made by Stålmark [182]: this restriction is not applicable for his proof. But there is another restriction in his case: it is crucial that negation is present in the language and the specific rules for it are used, in particular the following rule has to be in the system (we give it in two forms, with negation (on the left), and with implication together with falsum constant (in the centre), it is well-known that $\neg A$ can be defined as $A \to \bot$; the right-most rule is Peirce rule, which we will discuss a bit later and which is in a sense a more general rule: instead of $\bot$ it has an arbitrary formula B):

$$\dfrac{\begin{array}{c}[\neg A]\\ \mathfrak{D}\\ B\end{array}}{B} \qquad \dfrac{\begin{array}{c}[A \to \bot]\\ \mathfrak{D}\\ B\end{array}}{B} \qquad \dfrac{\begin{array}{c}[A \to B]\\ \mathfrak{D}\\ B\end{array}}{B}$$

Zimmermann [199] developed another proof: he used Peirce's rule, and it is not necessary for him to have a negation in the language, but the presence of an implication and Peirce's rule has become necessary. At that point, the choice of other connectives is flexible.

In addition to these restrictions on the languages, the above-mentioned proofs have one more disadvantage: they do not lead to the standard subformula property. Both of these issues were improved by Kürbis [103]: he considers the language with negation, conjunction, disjunction, and implication (later on, an existential quantifier has been added, but there are some difficulties with the general quantifier), and that any of these connectives can be removed from the language without any troubles, as well as the standard subformula property has been established. The price for these advantages (and at the same time, what made them possible) is the usage of general introduction and elimination rules. Notice that Kürbis uses a natural system with such rules developed by Milne [125]. So it seems reasonable to choose this system for consideration and extend it by modalities. However, we can go even further. In a recent paper [62], Geuvers and Hurkens prove the normalisation theorem and established the subformula property for classical propositional logic formulated in the language with at least one n-ary Boolean connective. The proof is obtained with the help of λ- and μ-abstractions. One of the considered in [62] natural deduction system is originally due to Segerberg [173]. Because of its generality we would like to consider this system. Moreover, we are going to present our own of the normalisation theorem for it: we do not use λ- and μ-abstractions, but generalise Kürbis' proof from [103].

After that, we add to Segerberg's system the modified versions of Biermann and de Paiva's [16] rules for $\Box$ and $\Diamond$ (although, unfortunately, not together but separately, since the rules were originally invented for intuitionistic modal logic and they block the proofs of the formulas $\Box A \leftrightarrow \neg\Diamond\neg A$ and $\Diamond A \leftrightarrow \neg\Box\neg A$) and prove normalisation for **S4** and **S5** in the language with at least one Boolean n-ary connective and either $\Box$ and $\Diamond$. Then we extend this result for the case of negative modalities: $\sim$ and $\dot{\sim}$. Finally, we discuss the difficulties that arise during our attempts to formulate natural deduction rules for the other non-standard modalities.

The *normalisation theorem* for a natural deduction system is a statement that any deduction in it can be transformed into one in a *normal form*, that is a deduction such that "(i) it contains no *maximal formula*, that is a formula that is the conclusion of an introduction rule and the major premise of an elimination rule (for its main connective); and (ii) no *maximal segment*, that is a sequence of formulas of the same shape arising from the applications of certain rules the last of which is major premise of an elimination rule." [104, p. 14224] The usage of general introduction rules requires some changes in this framework: as suggested in [104], maximal formula is understood as the major premise of a general elimination rule that is the *major assumption* discharged by an application of a general introduction rule (the major assumption of a general introduction rule for a connective $*$ is the assumption with a formula with $*$ as its main operator which is discharged by this rule). Here we need to emphasize one important feature of general introduction rules: "The introduction rules for a connective $*$ are formulated in terms of the discharge of assumptions of the form $A*B$, and every rule of the system is one that allows the derivation of an arbitrary formula from side-deductions of that formula and some further premises, as is the case with disjunction elimination in Gentzen's system." [104, p. 14224] This assumption $A*B$ is the major one, in contrast to some other assumptions, minor ones, which the rule may have. We would like to give one more important quotation from [104] clarifying the nature of the rules we are dealing with:

> "The difference between introduction and elimination rules lies in whether a formula with the connective governed by the rules as main operator is discharged above a side-deduction required for an application of the rule or whether such a formula is a premise of the rule." [104, p. 14224–14225]

One of the important consequences of the normalisation theorem (and shape of the rules) is the *subformula property*: a deduction has it iff "any formula that occurs on it is a subformula of either an undischarged assumption or of the conclusion." [104, p. 14224]

As we said above, we are going to prove normalisation for Segerberg's natural deduction system. This calculus is based on the idea of a 1-1 correspondence between inference rules and truth table entries. A rule is sound if and only if the connective in question has such an entry. Each entry has its own corresponding rule (or several rules). How does one obtain a sound and complete natural deduction system for a connective? Take as inference rules all the rules that correspond to all its entries, that is, those that are sound iff this connective has these entries.

The idea of using 1-1 correspondence between truth table entries and inference rules later on was independently rediscovered at least twice: first by Kooi and Tamminga [96, 189] for three-valued logics, and then by Geuvers and Hurkens [60] for classical logic (in [60] exactly Segerberg's system is reopened) and intuitionistic logic (despite of the fact that it is a non-tabular logic and the method at the first glance seems to be suitable only for tabular logics). Geuvers and Hurkens [61] presented a normalisation proof for their system for intuitionistic logic, but left the classical case untouched.

Let us consider a propositional language $\mathscr{L}_{(\circledcirc)_m}$ with the alphabet $\langle \mathcal{P}, \circledcirc_1, \ldots, \circledcirc_m, (,) \rangle$, where $\mathcal{P}$ is the set of propositional variables $\{p_1, p_2, \ldots\}$, $m \geqslant 1$, $\circledcirc_i$ $(1 \leqslant i \leqslant m)$ is an n-ary connective for $n \geqslant 1$. The set $\mathscr{F}_{(\circledcirc)_m}$ of all $\mathscr{L}_{(\circledcirc)_m}$-formulas, respectively, is defined inductively in a standard way.

Let us introduce the following notational convention which is in need to formulate Segerberg's inference rules.

Notation 42. Consider the set of natural numbers $\mathfrak{s} = \{1, \ldots, n\}$. By a 2-partitioning of $\mathfrak{s}$ we mean an ordered pair $\langle I, J \rangle$ such that $I \cup J = \mathfrak{s}$ and $I \cap J = \emptyset$.

For a better understanding of the relation of Segerberg's rules and truth tables, it is useful to apply also the subsequent notational convention.

Notation 43. In what follows, we are going to consider a partitioning of the following type: $\langle \mathfrak{t}, \mathfrak{f} \rangle$, where $\mathfrak{t} = \{i \in \mathfrak{s} \mid v(A_i) = 1, A_i \in \mathscr{F}_{(\circledcirc)_m}\}$ and $\mathfrak{f} = \{j \in \mathfrak{s} \mid v(A_j) = 0, A_j \in \mathscr{F}_{(\circledcirc)_m}\}$.

Besides, the following notation, which is rather in the spirit of Kooi and Tamminga [96, 189], is appropriate for the calculi in question.

Notation 44. The expression $f_{\circledcirc}(x_1, \ldots, x_n) = y$, where $x_1, \ldots, x_n, y \in \{1, 0\}$, means that if $v(A_1) = x_1, \ldots, v(A_n) = x_n$, then $v(\circledcirc(A_1, \ldots, A_n)) = y$, for each valuation v and all formulas $A_1, \ldots, A_n$. The expression $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{f}\rangle) = x$, where $x \in \{1, 0\}$, means that if $v(A_i) = 1$ (for each $i \in \mathfrak{t}$), and $v(A_j) = 0$ (for each $j \in \mathfrak{f}$), then $v(\circledcirc(A_1, \ldots, A_n)) = x$, for each valuation v.

Let us give the following definition from [145, Definition 4.1] restricted to the two-valued case (which is a slightly generalized version of Kooi and Tamminga's definition [96, Definition 2.1]) which perfectly suits for Segerberg's approach.

Definition 45 (Generalized single entry correspondence)**.** Let $x_1, \ldots, x_n, y \in \{1, 0\}$ and $A_1, \ldots, A_h$, $B_1, \ldots, B_g, C_1, \ldots, C_t \in \mathscr{F}_{\circledcirc(m)}$. Let E be a truth table (or matrix) entry of the type $f_{\circledcirc}(x_1, \ldots, x_n) = y$. Let $1 \leqslant l \leqslant h$ and I_l/A_l be an inference scheme of the type $B_1, \ldots, B_g, C_1 \vdash A_l, \ldots, C_t \vdash A_l/A_l$ or $C_1 \vdash A_l, \ldots, C_t \vdash A_l/A_l$. Then E is *characterised* by inference schemes $I_1/A_1, \ldots, I_h/A_h$, if

$$E \text{ if and only if } I_1 \models A_1, \ldots, I_k \models A_k.$$

By *connectives* we mean $\neg$ and $\circledcirc$. By *atomic formulas* we understand propositional variables. By *literals* we mean propositional variables and their negations. We consider tree-style deductions which have (un)discharged assumptions at the leaves and the conclusion at the root. We follow the policy regarding assumptions and vacuous discharge used in [104]:

> "Assumptions are assigned *assumption classes*, (at most) one for each assumption, marked by a natural number, different numbers for different assumption classes. Formula occurrences of different types must belong to different assumption classes. Formula occurrences of the same type may, but do not have to, belong to the same assumption class. Discharge of assumptions is marked by a square bracket around the formula: $[A]^i$, i being the assumption class to which A belongs, with the same label also occurring at the application of the rule at which the assumption is discharged. Assumption classes are chosen in such a way that if one assumption of an assumption class is discharged by an application of a rule, then it discharges all assumptions in that assumption class. Empty assumption classes are permitted: they are used in vacuous discharge, when a rule that allows for the discharge of assumptions is applied with no assumptions being discharged." [104, p. 14229]

Let us formulate Segerberg's rules [173, p. 558]:

$$R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{f}\rangle 1\big)^{a,b}\ \frac{\begin{array}{c}[\circledcirc(A_1, \ldots, A_n)]^a\\ \mathfrak{D}_1\\ B\end{array} \qquad \begin{array}{c}\\ \mathfrak{D}_2^{\dagger}\\ A_i^{\dagger}\end{array} \quad \begin{array}{c}[A_j^*]^b\\ \mathfrak{D}_3^*\\ B\end{array}}{B}$$

† for each $i \in \mathfrak{t}$, * for each $j \in \mathfrak{f}$.

$$R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{f}\rangle 0\big)^{a}\ \frac{\begin{array}{c}\mathfrak{D}_1\\ \circledcirc(A_1, \ldots, A_n)\end{array} \quad \begin{array}{c}\mathfrak{D}_2^{\dagger}\\ A_i^{\dagger}\end{array} \quad \begin{array}{c}[A_j^*]^a\\ \mathfrak{D}_3^*\\ B\end{array}}{B}$$

† for each $i \in \mathfrak{t}$, * for each $j \in \mathfrak{f}$.

As Segerberg notes [173, p. 558], each rule has $i + j + 1$ premises and 1 conclusion.

Let us denote Segerberg's natural deduction system for classical propositional logic via $\mathbb{ND}_{\mathbf{CPL}}$. In what follows, we write quite often $\circledcirc(\vec{A})$ for $\circledcirc(A_1, \ldots, A_n)$.

Definition 46 (Deduction in $\mathbb{ND}_{\mathbf{CPL}}$)**.**

1. The formula occurrence A is a deduction in $\mathbb{ND}_{\mathbf{CPL}}$ of A from the undischarged assumption A.

2. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are deductions in $\mathbb{ND}_{\mathbf{CPL}}$, then the applications of the above-mentioned rules are deductions of B in $\mathbb{ND}_{\mathbf{CPL}}$ from the undischarged assumptions in $\mathfrak{D}_1$ and $\mathfrak{D}_2$ apart from those in the assumption classes a and b, which are discharged.

3. Nothing else is a deduction in $\mathbb{ND}_{\mathbf{CPL}}$.

We write $\Gamma \vdash_{\mathbb{ND}_{\mathbf{CPL}}} A$ (or just $\Gamma \vdash_{\mathbf{CPL}} A$) if there is a deduction in $\mathbb{ND}_{\mathbf{CPL}}$ of (the formula occurrence) A from (occurrences of) some of the formulas in Γ.

Let us give some examples of the rules (the case when $\circledcirc$ is Boolean negation, denoted as $\neg$):

$$R_\neg\big(\langle\emptyset,\{1\}\rangle 1\big)^{a,b}\ \dfrac{\begin{matrix}[\neg A]^a & [A]^b\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ B & B\end{matrix}}{B} \qquad R_\neg\big(\langle\{1\},\emptyset\rangle 0\big)\ \dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ \neg A & A\end{matrix}}{B}$$

The rule $R_\neg\big(\langle\emptyset,\{1\}\rangle 1\big)$ is valid iff $f_\neg(0)=1$, the rule $R_\neg\big(\langle\{1\},\neg\rangle 0\big)$ is valid iff $f_\neg(1)=0$. As follows from Segerberg's results, these two rules form a sound and complete natural deduction system for the negation fragment of classical logic. In what follows, we denote the rule $R_\neg\big(\langle\emptyset,\{1\}\rangle 1\big)$ via (EM) and the rule $R_\circledcirc\big(\langle\{1\},\neg\rangle 0\big)$ via (EFQ).

One more example of the rules (the case when $\circledcirc$ is Boolean implication denoted as $\rightarrow$):

$$\begin{matrix}R_\rightarrow\big(\langle\emptyset,\{1,2\}\rangle 1\big)^{a,b,c}\\ f_\rightarrow(0,0)=1\end{matrix}\ \dfrac{\begin{matrix}[A_1\rightarrow A_2]^a & [A_1]^b & [A_2]^c\\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3\\ B & B & B\end{matrix}}{B} \qquad \begin{matrix}R_\rightarrow\big(\langle\{1\},\{2\}\rangle 1\big)^{a,b}\\ f_\rightarrow(0,1)=1\end{matrix}\ \dfrac{\begin{matrix}[A_1\rightarrow A_2]^a & [A_1]^b & \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3\\ B & B & A_2\end{matrix}}{B}$$

$$\begin{matrix}R_\rightarrow\big(\langle\{2,1\},\emptyset\rangle 1\big)^{a}\\ f_\rightarrow(1,1)=1\end{matrix}\ \dfrac{\begin{matrix}[A_1\rightarrow A_2]^a & & \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3\\ B & A_1 & A_2\end{matrix}}{B} \qquad \begin{matrix}R_\rightarrow\big(\langle\{1\},\{2\}\rangle 0\big)^{a}\\ f_\rightarrow(1,0)=0\end{matrix}\ \dfrac{\begin{matrix} & & [A_2]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3\\ A_1\rightarrow A_2 & A_1 & B\end{matrix}}{B}$$

These four rules form a sound and complete natural deduction system for the implication fragment of classical logic. Together with the above-mentioned two rules for negation, they form a sound and complete natural deduction system for the whole classical propositional logic formulated in the language with negation and implication.

Notice that a more convenient version of the rules for the implication fragment of classical logic is as follows (see e.g. [199]):

$$(\rightarrow I)^a\ \dfrac{\begin{matrix}[A_1]^a\\ \mathfrak{D}_1\\ A_2\end{matrix}}{A_1\rightarrow A_2} \qquad (MP)\ \dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ A_1\rightarrow A_2 & A_1\end{matrix}}{A_2} \qquad (P)^a\ \dfrac{\begin{matrix}[A_1\rightarrow A_2]^a\\ \mathfrak{D}\\ A_1\end{matrix}}{A_1}$$

General introduction/elimination rules for the implication fragment of classical logic are as follows (used by Milne [125], (TR) is Tarski's rule):

$$(\rightarrow I)^a\ \dfrac{\begin{matrix} & [A_1\rightarrow A_2]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ A_2 & B\end{matrix}}{B} \qquad (\mathrm{TR})^{a,b}\ \dfrac{\begin{matrix}[A_1]^a & [A_1\rightarrow A_2]^b\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ B & B\end{matrix}}{B} \qquad (\rightarrow E)^a\ \dfrac{\begin{matrix} & & [A_2]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3\\ A_1\rightarrow A_2 & A_1 & B\end{matrix}}{B}$$

The following theorem confirms the fact that there is a 1-1 correspondence between truth tables entries and inference rules in Segerberg's natural deduction system.

Theorem 47. *Let* **L** *be* **CPL** *formulated in the language* $\mathscr{L}_{(\circledcirc)_m}$. *Then*:

(1) $f_\circledcirc(\langle \mathfrak{t}, \mathfrak{f}\rangle) = 0$ *iff* $R_\circledcirc(\langle \mathfrak{t}, \mathfrak{f}\rangle 0)$ *is sound in* **L**.

(2) $f_\circledcirc(\langle \mathfrak{t}, \mathfrak{f}\rangle) = 1$ *iff* $R_\circledcirc(\langle \mathfrak{t}, \mathfrak{f}\rangle 1)$ *is sound in* **L**.

Proof. Follows from Segerberg's results [173]. □

Let us present an adaptation of [104, Definition 3] for our case.

Definition 48 (Terminology for Premises and Discharged Assumptions)**.**

1. In applications of the general elimination rules, formula occurrences taking the places of $\circledcirc(\vec{A})$ (rule $R_\circledcirc(\langle \mathfrak{t}, \mathfrak{f}\rangle 0)$) to the very left above the line are the *major premises*; formula occurrences taking the places of B at the end of subdeductions are the *arbitrary premises*; formula occurrences taking the place of A_i in applications of $R_\circledcirc(\langle \mathfrak{t}, \mathfrak{f}\rangle 0)$ are the *minor premises.*

2. In applications of the general introduction rules, formula occurrences taking the place of A_j in applications of $R_\circledcirc(\langle \mathfrak{t}, \mathfrak{f}\rangle 1)$ are the *specific premises*; formula occurrences taking the places of B at the end of subdeductions are the *arbitrary premises*; formula occurrences taking the places of the discharged assumptions $\circledcirc(\vec{A})$ (rule $R_\circledcirc(\langle \mathfrak{t}, \mathfrak{f}\rangle 1)$) are *the major assumptions* discharged by applications of the respective rules; formula occurrences taking the place of the discharged assumptions A_j in $R_\circledcirc(\langle \mathfrak{t}, \mathfrak{f}\rangle 1)$ are the *minor assumptions* discharged by applications of the respective rules.

Definition 49 (Maximal formula)**.** A *maximal formula* with the main operator $\circledcirc$ in a deduction in **CPL** is an occurrence of a formula $\circledcirc(A_1, \dots, A_n)$ that is the major premise of an application of a general elimination rule for $\circledcirc$ and the major assumption discharged by an application of a general introduction rule for $\circledcirc$.

Following [104, 103], we understand the notions of a segment, its length and degree, and a maximal segment in the subsequent way.

Definition 50. [Degree of a formula] We define the degree d of a formula A inductively as follows, where p is an atomic formula:

- $d(p) = 1$,
- $d(\circledcirc(A_1, \dots, A_n)) = \sum_{i=1}^{i=n} d(A_i) + 1$.

Definition 51 (Segment)**.** A *segment* is a sequence of formula occurrences $C_1 \dots C_h$ of the same shape in a deduction such that either

- $h > 1$, for all $g < h$, C_g is an arbitrary premise of an application of a rule and C_{g+1} is its conclusion, and C_h is not an arbitrary premise of an application of a rule,

- or $h \geq 1$ and C_1 is the conclusion of a general elimination rule and for all $g < h$, C_g is an arbitrary premise of an application of a rule and C_{g+1} is its conclusion, and C_h is not an arbitrary premise of an application of a rule.

The second clause of the definition of the segment is a generalisation of the second clause of [103, Definition 5]: there the rule (EFQ) (in the notation of [103], $\neg E$) is mentioned. We give a more general formulation, because among the special cases of the rule $R_\circledcirc(\langle \mathfrak{t}, \mathfrak{f}\rangle 0)$, there are not only the rule (EFQ), but also other rules like it. In general, they have the following shape (the case when $\mathfrak{f} = \emptyset$):

$$R_{\circledcirc}\big(\langle \mathfrak{t},\emptyset\rangle 0\big)\ \dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2^{\dagger}\\ \circledcirc(A_1,\ldots,A_n) & A_i^{\dagger}\end{matrix}}{B}$$

† for each $i \in \mathfrak{t}$.

Definition 52 (Length and degree of a segment)**.** The *length of a segment* is the number of formula occurrences of which it consists, its *degree* is the degree of any such formula. Since $C_1 \ldots C_n$ are all of the same shape, we will speak of the formula (as a type) constituting the segment.

Definition 53 (Maximal segment)**.** A *maximal segment* is a segment the last formula of which is the major premise of an elimination rule.

Definition 54 (Normal form)**.** A deduction is in *normal form* iff it contains neither maximal formulas nor maximal segments.

Definition 55 (Rank of Deductions)**.** The rank of a deduction $\mathfrak{D}$ is the pair $\langle d, l\rangle$, where d is the highest degree of a maximal formula or a maximal segment in $\mathfrak{D}$ or 0 if there is none, and l is the sum of the sum of the lengths of maximal segments of the highest degree and the number of maximal formulas of the highest degree in $\mathfrak{D}$. $\langle d, l\rangle < \langle d', l'\rangle$ iff either (i) $d < d'$ or (ii) $d = d$ and $l < l'$.

We begin our proof with an example of simplicity conversions which can remove applications of rules with vacuous discharge above arbitrary premises. Since these conversions are rather obvious, we give just one example.

$$R_{\circledcirc}\big(\langle \mathfrak{t},\mathfrak{f}\rangle 0\big)\ \dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2^{\dagger} & \mathfrak{D}_3^{\ddagger}\\ \circledcirc(\vec{A}) & A_i^{\dagger} & B^{\ddagger}\end{matrix}}{B} \quad \dashrightarrow \quad \begin{matrix}\mathfrak{D}_3^{\ddagger}\\ B^{\ddagger}\end{matrix}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{f}$.

In what follows, we suppose that if, after the application of a reduction procedure, applications of rules with vacuous discharge above arbitrary premises are obtained, then simplicity conversions are immediately applied.

Also, we would like to cover here the case when a maximal segment is generated by the rule $R_{\circledcirc}\big(\langle \mathfrak{t},\mathfrak{f}\rangle 0\big)$ such that $\mathfrak{f} = \emptyset$ (the second clause of Definition 51).

$$R_{\circledcirc}\big(\langle \mathfrak{t}',\emptyset\rangle 0\big)\ \dfrac{\begin{matrix} R_{\circledcirc}\big(\langle \mathfrak{t},\emptyset\rangle 0\big)\ \dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2^{\dagger}\\ \circledcirc(\vec{A}) & A_i^{\dagger}\end{matrix}}{\circledcirc(\vec{B})} & \begin{matrix}\mathfrak{D}_3^{\ddagger}\\ B_i^{\ddagger}\end{matrix}\end{matrix}}{C} \quad \dashrightarrow \quad R_{\circledcirc}\big(\langle \mathfrak{t},\emptyset\rangle 0\big)\ \dfrac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2^{\dagger}\\ \circledcirc(\vec{A}) & A_i^{\dagger}\end{matrix}}{C}$$

† for each $i \in \mathfrak{t}$, ‡ for each $i \in \mathfrak{t}'$.

Case 1. The maximal formula $\circledcirc(\vec{A})$ produced by applications of the rules $R_{\circledcirc}\big(\langle \mathfrak{t},\mathfrak{f}\rangle 0\big)$ and $R_{\circledcirc}\big(\langle \mathfrak{t}',\mathfrak{f}'\rangle 1\big)$ which we denote by $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively.

$$\mathfrak{R}_2^{a,c}\ \dfrac{\begin{matrix}\mathfrak{R}_1^{b}\ \dfrac{\begin{matrix} & & [A_j^{*}]^{b}\\ & \mathfrak{D}_1^{\dagger} & \mathfrak{D}_2^{*}\\ [\circledcirc(\vec{A})]^{a} & A_i^{\dagger} & B\end{matrix}}{B} & & [A_{j'}^{\star}]^{c}\\ \mathfrak{H} & \mathfrak{D}_3^{\S} & \mathfrak{D}_4^{\star}\\ C & A_{i'}^{\S} & C\end{matrix}}{C}$$

†for each $i \in \mathfrak{t}$, *for each $j \in \mathfrak{f}$; §for each $i' \in \mathfrak{t}'$, *for each $j' \in \mathfrak{f}'$.

Let us recall that all the formulas $A_i, A_j, A_{i'}, A_{j'}$, are subformulas of $\circledcirc(\vec{A}) = \circledcirc(A_1, \dots, A_n)$. We need to have more details about the other rules which are present in the system. Let us observe that this case can be reformulated in the following way.

$$\mathfrak{R}_2^{a,c}\,\dfrac{\begin{array}{c}\mathfrak{R}_1^{b}\,\dfrac{[\circledcirc(\vec{A})]^a \qquad X_1 \qquad \dots \qquad X_n}{B}\\ \mathfrak{H}\\ C\end{array} \qquad Y_1 \qquad \dots \qquad Y_n}{C}$$

where $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are abbreviations for the following derivations, for any $t, u \in \{1, \dots, n\}$.

$$X_t = \begin{cases} \begin{array}{c}[A_t]^b\\ \mathfrak{D}_2\\ B\end{array} & \textit{iff} \quad t \in \mathfrak{f};\\[2ex] \begin{array}{c}\mathfrak{D}_1\\ A_t\end{array} & \textit{iff} \quad t \in \mathfrak{t}; \end{cases} \qquad Y_u = \begin{cases} \begin{array}{c}[A_u]^c\\ \mathfrak{D}_4\\ C\end{array} & \textit{iff} \quad u \in \mathfrak{f}';\\[2ex] \begin{array}{c}\mathfrak{D}_3\\ A_u\end{array} & \textit{iff} \quad u \in \mathfrak{t}'. \end{cases}$$

As follows from these equalities and soundness of our natural deduction systems, there is $l \in \{1, \dots, n\}$ such that $X_l \neq Y_l$.[14] Then the following combinations are possible (we present them in the form of ordered pairs $\langle X_l, Y_l \rangle$):

$$\mathcal{C}_1 = \left\langle \begin{array}{c}[A_l]^b\\ \mathfrak{D}_2\\ B\end{array}, \begin{array}{c}\mathfrak{D}_3\\ A_l\end{array} \right\rangle \qquad \mathcal{C}_2 = \left\langle \begin{array}{c}\mathfrak{D}_1\\ A_l\end{array}, \begin{array}{c}[A_l]^c\\ \mathfrak{D}_4\\ C\end{array} \right\rangle$$

Let us consider the case $\mathcal{C}_1$.

$$\mathfrak{R}_2^{a,c}\,\dfrac{\begin{array}{c}\mathfrak{R}_1^{b}\,\dfrac{[\circledcirc(\vec{A})]^a \qquad X_1 \dots X_{l-1} \qquad \begin{array}{c}[A_l]^b\\ \mathfrak{D}_2\\ B\end{array} \qquad X_{l+1} \dots X_n}{B}\\ \mathfrak{H}\\ C\end{array} \qquad Y_1 \dots Y_{l-1} \qquad \begin{array}{c}\mathfrak{D}_3\\ A_l\end{array} \qquad Y_{l+1} \dots Y_n}{C}$$

Then we can introduce the following reduction procedures:

$$\begin{array}{c}\mathfrak{D}_3\\ A_l\\ \mathfrak{D}_2\\ B\\ \mathfrak{H}\\ C\end{array} \qquad\qquad \mathfrak{R}_2^{a,c}\,\dfrac{\begin{array}{c}\mathfrak{D}_3\\ A_l\\ \mathfrak{D}_2\\ B\\ \mathfrak{H}\\ C\end{array} \qquad Y_1 \dots Y_{l-1} \qquad \begin{array}{c}\mathfrak{D}_3\\ A_l\end{array} \qquad Y_{l+1} \dots Y_n}{C}$$

[14] Otherwise $\mathfrak{t} = \mathfrak{t}'$, and $\mathfrak{f} = \mathfrak{f}'$ which implies that both $f_\circledcirc(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 0$ and $f_\circledcirc(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 1$.

If the assumption class a contains one formula, the reduction procedure presented on the left should be applied. If there are more than one formula in the assumption class a, then in order to be sure that all of them are discharged we apply the rule $\mathfrak{R}_2$ at the end, that is we use the reduction procedure presented on the right. In what follows, in similar cases we will display only reduction procedures with such an application of a rule at the end and do not comment it, assuming by default that such an application is needed only if there are undischarged formulas.

The case $\mathcal{C}_2$ seems to be similar, but there is a principal difference, let us consider it.

$$
\mathfrak{R}_2^{a,c}\dfrac{\begin{array}{c} \mathfrak{R}_1^{b}\dfrac{[\circledcirc(\vec{A})]^a \quad X_1\ldots X_{l-1} \quad \begin{array}{c}\mathfrak{D}_1\\ A_l\end{array} \quad X_{l+1}\ldots X_n}{B} \\ \mathfrak{H} \\ C \end{array} \qquad Y_1\ldots Y_{l-1} \qquad \begin{array}{c}[A_l]^c\\ \mathfrak{D}_4\\ C\end{array} \qquad Y_{l+1}\ldots Y_n}{C}
$$

At the first glance we should use the following reduction procedures:

$$
\begin{array}{c}\mathfrak{D}_1\\ A_l\\ \mathfrak{D}_4\\ C\end{array}
\qquad\qquad
\mathfrak{R}_2^{a,c}\dfrac{\begin{array}{c}\mathfrak{D}_1\\ A_l\\ \mathfrak{D}_4\\ C\end{array} \qquad Y_1\ldots Y_{l-1} \qquad \begin{array}{c}\mathfrak{D}_1\\ A_l\end{array} \qquad Y_{l+1}\ldots Y_n}{C}
$$

However, there might be a problem, if we have the following situation:

$$
\begin{array}{c}\mathfrak{D}_1\\ A_l\end{array} \quad \text{has the form} \quad \begin{array}{c}[D]^e\\ \mathfrak{D}_1'\\ A_l\end{array}
$$

and the assumption class e is discharged in $\mathfrak{H}$. According to these reductions, we do not have $\mathfrak{H}$ anymore, thus we cannot discard e. Let us illustrate such a troublesome situation by an example from [103, p. 117]. The rules for negation used in [103] are just the same as the ones produced by Segerberg's method. Let us have a natural deduction with the rules (EM) and (EFQ) as well as the rule corresponding to the entry $f_\circ(0,0) = 0$ for some connective $\circ$ (it can be $\vee$, as in the example from [103]).

$$
{}_{3,4}\dfrac{\begin{array}{c}[A]^3\\ \mathfrak{E}_1\\ C\end{array} \qquad {}_{1,2}\dfrac{\begin{array}{c}\mathfrak{H}_1\\ D\vee E\end{array} \qquad \begin{array}{c}(\mathrm{EFQ})\dfrac{[\neg A]^4 \quad \begin{array}{c}[D]^1\\ \mathfrak{E}_2\\ A\end{array}}{B}\\ \mathfrak{H}_2\\ C\end{array} \qquad \begin{array}{c}[E]^2\\ \mathfrak{H}_3\\ C\end{array}}{C}}{C}
\qquad\qquad
\begin{array}{c}[D]^1\\ \mathfrak{E}_2\\ [A]\\ \mathfrak{E}_1\\ C\end{array}
$$

If we transform the deduction displayed on the left according to the procedure displayed on the right, then we can't discharge assumption class 1. Thus, we need to have a different reduction procedure:

$$
{}_{1,2}\dfrac{\begin{array}{c}\mathfrak{H}_1\\ D\vee E\end{array} \qquad \begin{array}{c}[D]^1\\ \mathfrak{E}_2\\ [A]\\ \mathfrak{E}_1\\ C\end{array} \qquad \begin{array}{c}[E]^2\\ \mathfrak{H}_3\\ C\end{array}}{C}
$$

Coming back to our case, we can introduce the following reduction procedure:

$$\mathfrak{R}_2^{a,c}\ \dfrac{\begin{array}{c}\mathfrak{D}_1\\ A_l\\ \mathfrak{D}_4\\ C\\ \mathfrak{H}^*\\ C\end{array} \qquad Y_1\ldots Y_{l-1} \qquad \begin{array}{c}\mathfrak{D}_1\\ A_l\end{array} \qquad Y_{l+1}\ldots Y_n}{C}$$

where $\mathfrak{H}^*$ is defined as follows (for the case of the rule with one or two arbitrary premisses, it can be easily generalised to the case of the rules with n arbitrary premisses):

> "Let $\rho_1 \ldots \rho_m$ be the sequence of applications of rules in $\mathfrak{H}$ that discharge assumptions in $\mathfrak{D}_1$ from top to bottom. Let ρ_1^* be an application of the same rule as ρ_1, with its major, minor and specific premises concluded as in ρ_1 and its arbitrary premises and conclusion replaced by C. If ρ_1^* has only one arbitrary premise, conclude it with the deduction ending in the upper C in the schematic representation of the reduction procedure. <...> If ρ_1^* has two arbitrary premises, then one is concluded as described previously, and to conclude the other, observe that in that case $\mathfrak{H}^*$ contains a subdeduction of C from the conclusion E of ρ_1: append it to the deduction concluding the other arbitrary premise of ρ_1 to conclude the other arbitrary premise of ρ_1^*, deleting redundant applications of rules (i.e., those discharging assumptions that do not stand above that arbitrary premise of ρ_1). Continue in the same way with ρ_2 until you reach ρ_m." [103, p. 116, the notation adjusted]

In what follows, we call such a construction of $\mathfrak{H}^*$ ρ-reduction.

Case 2. The maximal segment with the formula $\circledcirc_1(\vec{B})$ and the applications of the rules $R_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{f}\rangle 1\big)$ and $R_{\circledcirc_1}\big(\langle\mathfrak{t}',\mathfrak{f}'\rangle 0\big)$ which we denote by $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively.

$$\mathfrak{R}_2^{c}\ \dfrac{\mathfrak{R}_1^{a,b}\ \dfrac{\begin{array}{c}[\circledcirc_2(\vec{A})]^a\\ \mathfrak{D}_1\\ \circledcirc_1(\vec{B})\end{array} \qquad \begin{array}{c}\mathfrak{D}_2^{\dagger}\\ A_j^{\dagger}\end{array} \qquad \begin{array}{c}[A_k^*]^b\\ \mathfrak{D}_3^*\\ \circledcirc_1(\vec{B})\end{array}}{\circledcirc_1(\vec{B})} \qquad \begin{array}{c}\mathfrak{D}_4^{\S}\\ B_{j'}^{\S}\end{array} \qquad \begin{array}{c}[B_{k'}^{\star}]^c\\ \mathfrak{D}_5^{\star}\\ C\end{array}}{C}$$

†for each $i \in \mathfrak{t}$, *for each $j \in \mathfrak{f}$; §for each $i' \in \mathfrak{t}'$, *for each $j' \in \mathfrak{f}'$.

The permutative reduction procedure is as follows.

$$\mathfrak{R}_1^{a,b}\ \dfrac{\mathfrak{R}_2^{c}\ \dfrac{\begin{array}{c}[\circledcirc_2(\vec{A})]^a\\ \mathfrak{D}_1\\ \circledcirc_1(\vec{B})\end{array} \qquad \begin{array}{c}\mathfrak{D}_4^{\S}\\ B_{j'}^{\S}\end{array} \qquad \begin{array}{c}[B_{k'}^{\star}]^c\\ \mathfrak{D}_5^{\star}\\ C\end{array}}{C} \qquad \begin{array}{c}\mathfrak{D}_2^{\dagger}\\ A_j^{\dagger}\end{array} \qquad \mathfrak{R}_2^{c}\ \dfrac{\begin{array}{c}[A_k^*]^b\\ \mathfrak{D}_3^*\\ \circledcirc_1(\vec{B})\end{array} \qquad \begin{array}{c}\mathfrak{D}_4^{\S}\\ B_{j'}^{\S}\end{array} \qquad \begin{array}{c}[B_{k'}^{\star}]^c\\ \mathfrak{D}_5^{\star}\\ C\end{array}}{C}}{C}$$

†for each $i \in \mathfrak{t}$, *for each $j \in \mathfrak{f}$; §for each $i' \in \mathfrak{t}'$, *for each $j' \in \mathfrak{f}'$.

Notice that if C is also on a maximal segment, the procedure increases its length by one. But that can be handled by choosing a suitable maximal segment, i.e., ensuring that the one which have been shortened by the procedure, consists of formulas of a higher degree than C.

Case 3. The maximal segment with the formula $\circledcirc_1(\vec{B})$ and the applications of the rules $R_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{f}\rangle 0\big)$ and $R_{\circledcirc_1}\big(\langle\mathfrak{t}',\mathfrak{f}'\rangle 0\big)$ (we use for them the abbreviations $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively).

$$\mathfrak{R}_2^c\,\dfrac{\mathfrak{R}_1^b\,\dfrac{\begin{matrix}\mathfrak{D}_1\\ \circledcirc_2(\vec{A})\end{matrix}\quad \begin{matrix}\mathfrak{D}_2^\dagger\\ A_j^\dagger\end{matrix}\quad \begin{matrix}[A_k^*]^b\\ \mathfrak{D}_3^*\\ \circledcirc_1(\vec{B})\end{matrix}}{\circledcirc_1(\vec{B})}\quad \begin{matrix}\mathfrak{D}_4^\S\\ B_{j'}^\S\end{matrix}\quad \begin{matrix}[B_{k'}^\star]^c\\ \mathfrak{D}_5^\star\\ C\end{matrix}}{C}$$

†for each $i \in \mathfrak{t}$, *for each $j \in \mathfrak{f}$; §for each $i' \in \mathfrak{t}'$, *for each $j' \in \mathfrak{f}'$.

The permutative reduction procedure is as follows.

$$\mathfrak{R}_1^b\,\dfrac{\mathfrak{R}_2^c\,\dfrac{\begin{matrix}\mathfrak{D}_1\\ \circledcirc_2(\vec{A})\end{matrix}\quad \begin{matrix}\mathfrak{D}_4^\S\\ B_{j'}^\S\end{matrix}\quad \begin{matrix}[B_{k'}^\star]^c\\ \mathfrak{D}_5^\star\\ C\end{matrix}}{C}\quad \begin{matrix}\mathfrak{D}_2^\dagger\\ A_j^\dagger\end{matrix}\quad \mathfrak{R}_2^c\,\dfrac{\begin{matrix}[A_k^*]^b\\ \mathfrak{D}_3^*\\ \circledcirc_1(\vec{B})\end{matrix}\quad \begin{matrix}\mathfrak{D}_4^\S\\ B_{j'}^\S\end{matrix}\quad \begin{matrix}[B_{k'}^\star]^c\\ \mathfrak{D}_5^\star\\ C\end{matrix}}{C}}{C}$$

†for each $i \in \mathfrak{t}$, *for each $j \in \mathfrak{f}$; §for each $i' \in \mathfrak{t}'$, *for each $j' \in \mathfrak{f}'$.

Theorem 56. *Any deduction in* **CPL** *can be converted into a deduction in normal form.*

Proof. By induction over the rank of deductions. The structure of the proof is as follows: first we eliminate all maximal segments, then we eliminate all maximal formulas. Following Prawitz [161, p. 50], we require "to pick a maximal segment σ of highest degree such that (i) no maximal segment of highest degree stands above σ; (ii) no maximal segment of highest degree stands above or contains a formula side-connected with the last formula of σ." We have a similar requirement for maximal formulas[15]: pick a maximal formula F of highest degree such that (i) no maximal formula of highest degree stands above F; (ii) no maximal formula of highest degree stands above or contains a formula side-connected with F. □

Corollary 57. *If $\Gamma \vdash_{\mathbf{CPL}} A$, then there is a deduction in normal form with an occurrence of A as the conclusion and occurrences of the formulas in Γ as the undischarged assumptions.*

Theorem 58. *If $\mathfrak{D}$ is a deduction in normal form, then all major premises of elimination rules are (discharged or undischarged) assumptions of $\mathfrak{D}$.*

Proof. By the form of deductions in normal form, as a result of the permutative reduction procedures. □

Let us adopt [104, Definition 9] for our case.

Definition 59 (Branch)**.** A *branch* in a deduction is a sequence of formula occurrences $D_1 \dots D_h$ such that D_1 is an assumption of the deduction that is neither discharged by an elimination rule nor the major assumption discharged by an introduction rule, D_h is either the conclusion of the deduction or the minor premise of one of elimination rules, and for each $g < h$: if D_g is the major premise of an elimination rule, D_{g+1} is an assumption discharged by it; if D_g is the specific premise of an introduction rule, D_{g+1} is a major assumption discharged by it; and if D_g is an arbitrary premise (of an introduction or an elimination rule rule), D_{g+1} is the conclusion of the rule.

[15]According to Prawitz, a maximal formula is a maximal segment of the length 1, but since it is not the case in our terminology, we have to rephrase Prawitz's requirement for maximal segments for the case of maximal formulas.

Corollary 60. *If any major premises of elimination rules are on a branch in a deduction in normal form, then they precede any major assumptions discharged by introduction rules that are on the branch.*

Proof. Follows from Theorem 58. □

Definition 61 (Order of Branches)**.** A branch has *order* 0 if its last formula is the conclusion of the deduction; it has order $h+1$ if its last formula is the minor premise of an application of an elimination rule, the major premise of which is on a branch of order h. A branch of order 0 is also called a main branch in the deduction.

Definition 62 (Subformula Property)**.** A deduction $\mathfrak{D}$ of a conclusion C from the undischarged assumptions $A_1 ... A_h$ has the *subformula property* iff every formula on the deduction is a subformula either of C or of $A_1 ... A_h$.

Definition 63 (Negation Subformula Property)**.** A deduction $\mathfrak{D}$ of a conclusion C from the undischarged assumptions $A_1 ... A_h$ has the *negation subformula property* iff every formula on the deduction is a subformula or a negation of a subformula either of C or of $A_1 ... A_h$.

Theorem 64. *Deductions in normal forms in* **CPL** *have the subformula property.*

Proof. By inspection of the rules and an induction over the order of branches. □

Let us propose generalized introduction and elimination versions of Biermann and de Paiva's modal rules (we need them for unifying the modal case with the propositional one, which is based on this type of the rules):

$$(\Box GI)^{a,b}\ \frac{\begin{matrix}[\Box A]^a \\ \mathfrak{D}_0 \\ F\end{matrix} \quad \begin{matrix}\mathfrak{D}_1 \dots \mathfrak{D}_m \\ B_1 \dots B_m\end{matrix} \quad \begin{matrix}[B_1 \dots B_m]^b \\ \mathfrak{D} \\ A\end{matrix}}{F} \qquad (\Box GE)^a\ \frac{\begin{matrix}\mathfrak{D}_1 \\ \Box A\end{matrix} \quad \begin{matrix}[A]^a \\ \mathfrak{D}_2 \\ F\end{matrix}}{F}$$

$$(\Diamond GI)^a\ \frac{\begin{matrix}[\Diamond A]^a \\ \mathfrak{D}_1 \\ F\end{matrix} \quad \begin{matrix}\mathfrak{D}_2 \\ A\end{matrix}}{F} \qquad (\Diamond E)^a\ \frac{\begin{matrix}\mathfrak{D}_0 \\ \Diamond A\end{matrix} \quad \begin{matrix}\mathfrak{D}_1 \dots \mathfrak{D}_m \\ B_1 \dots B_m\end{matrix} \quad \begin{matrix}[A, B_1 \dots B_m]^a \\ \mathfrak{C} \\ C\end{matrix}}{C}$$

$B_1, \dots, B_m$ are exactly the undischarged assumptions in $\mathfrak{D}$ in $(\Box GI)$ and $A, B_1, \dots, B_m$ in $\mathfrak{C}$ in $(\Diamond E)$. For **S4**, $B_1, \dots, B_m$ are required to be of the form $\Box D_1, \dots, \Box D_m$ and C to be of the form $\Diamond D$. For **S5**, $A, B_1, \dots, B_m, C$ are required to be modalized. As we have already said above, Biermann and de Paiva's rules were originally developed for intuitionistic logic, so they can be used in classical case with some restrictions. One of them is the necessity to restrict the language: either $\Box$ or $\Diamond$ can be present in it, since it seems impossible to prove the formulas $\Box A \leftrightarrow \neg\Diamond\neg A$ and $\Diamond A \leftrightarrow \neg\Box\neg A$. In fact, in **S4** we can deal only with $\Box$, since the formulation of the rules for $\Diamond$ require the presence of $\Box$ in the language which we cannot afford to ourselves.

Let us give some examples of derivations in **S4** with the use of these rules as well as some above-mentioned rules for implication.

$$\dfrac{\dfrac{[\Box A]^a \qquad [A]^b}{A}\,(\Box GE)^b}{\Box A \to A}\,(\to I)^a \qquad\qquad \dfrac{\dfrac{[\Box\Box A]^a \qquad [\Box A]^b \qquad [\Box A]^c}{\Box\Box A}\,(\Box GI)^{a,c}}{\Box A \to \Box\Box A}\,(\to I)^b$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\Box B]^e \qquad [\Box(A\to B)]^g \qquad [\Box A]^f \qquad \dfrac{[\Box(A\to B)]^a \qquad [A\to B]^b}{A\to B}\,(\Box GE)^b \qquad \dfrac{[\Box A]^c \qquad [A]^d}{A}\,(\Box GE)^d}{B}\,(MP)}{\Box B}\,(\Box GI)^{a,c,e}}{\Box A\to\Box B}\,(\to I)^f}{\Box(A\to B)\to(\Box A\to\Box B)}\,(\to I)^g$$

Let us give an example of derivations in **S5** with the use of these rules as well as some above-mentioned rules for implication.

$$\dfrac{\dfrac{[\Box\neg\Box A]^a \qquad [\neg\Box A]^c \qquad [\neg\Box A]^b}{\Box\neg\Box A}\,(\Box GI)^{a,b}}{\neg\Box A\to\Box\neg\Box A}\,(\to I)^c$$

Theorem 65. *Let* $\mathbf{L}\in\{\mathbf{S4}^{\Box},\mathbf{S5}^{\Box},\mathbf{S5}^{\Diamond}\}$*. For any formula* A*,* A *is valid in* $\mathbf{L}$ *iff it is provable in the natural deduction system for* $\mathbf{L}$*.*

Proof. Soundness and completeness of the propositional part of the calculi in question follow from Segerberg's paper [173]. The soundness of the rules for $\Box$ and $\Diamond$ is known in intuitionistic **S4** and **S5** [16, 100]. Since intuitionistic **S4** and **S5** are sublogics of classical **S4** and **S5**, these rules are sound in the classical case as well. As for completeness, one needs to prove all the formulas and rules of a Hilbert-style formulation of classical **S4** and **S5** by natural deduction systems. The justification for the necessitation rule is shown by [16], the justification of axioms of **S4** and **S5** one may see above (in the language with $\Box$ only; one can similarly prove them in the language with $\Diamond$ only). $\Box$

As for normalisation, we need to consider the following cases.

Case 1. The maximal formula of the form $\Box A$ produced by the rules $(\Box GE)$ and $(\Box GI)$.

$$(\Box GI)^{a,c}\,\dfrac{\begin{matrix}(\Box GE)^b\,\dfrac{[\Box A]^a \qquad \begin{matrix}[A]^b\\ \mathfrak{E}_1\\ F\end{matrix}}{F}\\ \mathfrak{E}_3\\ G\end{matrix} \qquad \begin{matrix}\mathfrak{D}_1\ldots\mathfrak{D}_m\\ B_1\ldots B_m\end{matrix} \qquad \begin{matrix}[B_1\ldots B_m]^c\\ \mathfrak{E}_2\\ A\end{matrix}}{G}$$

We transform this derivation into the following one.

$$(\Box GI)^{a,c}\,\dfrac{\begin{matrix}\mathfrak{D}_1\ldots\mathfrak{D}_m\\ B_1\ldots B_m\\ \mathfrak{E}_2\\ A\\ \mathfrak{E}_1\\ F\\ \mathfrak{E}_3\\ G\end{matrix} \qquad \begin{matrix}\mathfrak{D}_1\ldots\mathfrak{D}_m\\ B_1\ldots B_m\end{matrix} \qquad \begin{matrix}[B_1\ldots B_m]^c\\ \mathfrak{E}_2\\ A\end{matrix}}{G}$$

Case 2. The maximal formula of the form $\Diamond A$ produced by the rules $(\Diamond E)$ and $(\Diamond GI)$.

$$(\Diamond GI)^a\,\dfrac{\begin{matrix}(\Diamond E)^b\,\dfrac{[\Diamond A]^a \qquad \begin{matrix}\mathfrak{D}_1\ldots\mathfrak{D}_m\\ B_1\ldots B_m\end{matrix} \qquad \begin{matrix}[A, B_1\ldots B_m]^b\\ \mathfrak{E}_1\\ C\end{matrix}}{C}\\ \mathfrak{E}_3\\ D\end{matrix} \qquad \begin{matrix}\mathfrak{E}_2\\ A\end{matrix}}{D} \quad \dashrightarrow \quad (\Diamond GI)^a\,\dfrac{\begin{matrix}\mathfrak{E}_2\ \mathfrak{D}_1\ldots\mathfrak{D}_m\\ A\ \ B_1\ldots B_m\\ \mathfrak{E}_1\\ C\\ \mathfrak{E}_3\\ G\end{matrix} \qquad \begin{matrix}\mathfrak{E}_2\\ A\end{matrix}}{G}$$

Case 3. The maximal segment with the formula $\circledcirc(\vec{F})$ produced by the rules $R(\langle\mathfrak{t},\mathfrak{f}\rangle 0)$ and $(\Box GE)$. $\mathfrak{R}_1$ and $\mathfrak{R}_2$ stand for $R(\langle\mathfrak{t},\mathfrak{f}\rangle 0)$ and $(\Box GE)$, respectively.

$$
\mathfrak{R}_2\,\dfrac{\mathfrak{R}_1\,\dfrac{\begin{matrix}\mathfrak{D}_0\\ \Box A\end{matrix}\qquad \begin{matrix}[A]^a\\ \mathfrak{D}_1\\ \circledcirc(\vec{F})\end{matrix}}{\circledcirc(\vec{F})}\qquad \begin{matrix}\mathfrak{D}_2^{\dagger}\\ F_j^{\dagger}\end{matrix}\qquad \begin{matrix}[F_k^*]^b\\ \mathfrak{D}_3^*\\ G\end{matrix}}{G}
\quad \dashrightarrow \quad
\mathfrak{R}_1\,\dfrac{\begin{matrix}\mathfrak{D}_0\\ \Box A\end{matrix}\qquad \mathfrak{R}_2\,\dfrac{\begin{matrix}[A]^a\\ \mathfrak{D}_1\\ \circledcirc(\vec{F})\end{matrix}\qquad \begin{matrix}\mathfrak{D}_2^{\dagger}\\ F_j^{\dagger}\end{matrix}\qquad \begin{matrix}[F_k^*]^b\\ \mathfrak{D}_3^*\\ G\end{matrix}}{G}}{G}
$$

†for each $i \in \mathfrak{t}$, *for each $j \in \mathfrak{f}$.

The cases with maximal segments produced by the rules $R(\langle\mathfrak{t},\mathfrak{f}\rangle 0)$ and $(\Box GI)$ as well as $R(\langle\mathfrak{t},\mathfrak{f}\rangle 0)$ and $(\Diamond GI)$ are considered similarly. Notice that the case $R(\langle\mathfrak{t},\mathfrak{f}\rangle 0)$ and $(\Diamond E)$ does not exists, since the formula C in the conclusion of the of the rule $(\Diamond E)$ is either $\Diamond D$ (in the case of **S4**) or a modalized formula (in the case of **S5**), but not the formula of the form $\circledcirc(\vec{F})$.

Case 4. The maximal segment with the formula $\Diamond A$ produced by two applications of the rule $(\Diamond E)$.

$$
{}_b\,\dfrac{\dfrac{\begin{matrix}\mathfrak{E}_0\\ \Diamond E\end{matrix}\qquad \begin{matrix}\mathfrak{E}_1\ldots\mathfrak{E}_l\\ F_1\ldots F_l\end{matrix}\qquad \begin{matrix}[E,F_1\ldots F_l]^a\\ \mathfrak{H}_1\\ \Diamond A\end{matrix}}{\Diamond A}\,{}_a\qquad \begin{matrix}\mathfrak{D}_1\ldots\mathfrak{D}_m\\ B_1\ldots B_m\end{matrix}\qquad \begin{matrix}[A,B_1\ldots B_m]^b\\ \mathfrak{H}_2\\ C\end{matrix}}{C}
$$

where $E, F_1 \ldots F_l$ and $A, B_1, \ldots, B_m$ are exactly the undischarged assumptions in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. For **S4**, B_i is required to be of the form $\Box D_i$, F_i is required to be of the form $\Box H_i$ $(1 \leqslant i \leqslant m)$, C to be of the form $\Diamond D$. For **S5**, $A, B_1, \ldots, B_m, C, E, F_1 \ldots F_l$ are required to be modalized.

We change the order of the applications of the rules as follows:

$$
\dfrac{\begin{matrix}\mathfrak{E}_0\\ \Diamond E\end{matrix}\qquad \begin{matrix}\mathfrak{E}_1\ldots\mathfrak{E}_l\\ F_1\ldots F_l\end{matrix}\qquad {}_b\,\dfrac{\begin{matrix}[E,F_1\ldots F_l]^a\\ \mathfrak{H}_1\\ \Diamond A\end{matrix}\qquad \begin{matrix}\mathfrak{D}_1\ldots\mathfrak{D}_m\\ B_1\ldots B_m\end{matrix}\qquad \begin{matrix}[A,B_1\ldots B_m]^b\\ \mathfrak{H}_2\\ C\end{matrix}}{C}\,{}_a}{C}
$$

where $E, F_1 \ldots F_l$ and $A, B_1, \ldots, B_m$ are exactly the undischarged assumptions in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. For **S4**, B_i is required to be of the form $\Box D_i$, F_i is required to be of the form $\Box H_i$ $(1 \leqslant i \leqslant m)$, C to be of the form $\Diamond D$. For **S5**, $A, B_1, \ldots, B_m, C, E, F_1 \ldots F_l$ are required to be modalized.

The cases with maximal segments produced by two applications of the rule $(\Box GI)$, two applications of the rule $(\Box GE)$, and two applications of the rule $(\Diamond GI)$ are considered similarly.

Theorem 66. *Any deduction in* **S4** *and* **S5** *can be converted into a deduction in normal form.*

Proof. By induction on the rank of deductions, using the above presented reduction, similarly to Theorem 56. □

As Biermann and de Paive [16] note, in order to establish the subformula property, some addition reductions are needed; they call them commuting conversions. Let us present these conversions, i.e., their adaptation for our formulation of Biermann and de Paive's rules. Let us consider the case of **S4**, the rules for $\Box$. The rule $(\Box GI)$ in **S4** is as follows:

$$\dfrac{\begin{matrix}[\Box A]^a\\ \mathfrak{D}_0\\ F\end{matrix} \qquad \begin{matrix}\mathfrak{D}_1\dots\mathfrak{D}_m\\ \Box D_1\dots\Box D_m\end{matrix} \qquad \begin{matrix}[\Box D_1\dots\Box D_m]^b\\ \mathfrak{D}\\ A\end{matrix}}{F}$$

As Biermann and de Paive [16] write, there is no guarantee that $\Box D_1 \dots \Box D_m$ are subformulas of of $\Box A$ or an undischarged assumption. There are two reasons for that: $\Box D_i$ can be a conclusion of some general elimination rule, and $\Box D_i$ can be obtained as a result of an application of $(\Box I)$. Since in our case all introduction rules are in a general form, we can add to the first clause that $\Box D_i$ can be the conclusion of some general introduction rule, except $(\Box GI)$. Let us consider these cases.

Case 1. $\Box D_i$ is a conclusion of some general elimination rule or some general introduction rule, except $(\Box GI)$.

$${}_{a,b}\dfrac{\begin{matrix}[\Box A]^a\\ \mathfrak{D}_0\\ F\end{matrix} \qquad \begin{matrix}\mathfrak{D}_1\dots\mathfrak{D}_{i-1}\\ \Box D_1\dots\Box D_{i-1}\end{matrix} \qquad \dfrac{\mathfrak{D}'_i \qquad \begin{matrix}\mathfrak{D}''_i\\ \Box D_i\end{matrix}}{\Box D_i} \qquad \begin{matrix}\mathfrak{D}_{i+1}\dots\mathfrak{D}_m\\ \Box D_{i+1}\dots\Box D_m\end{matrix} \qquad \begin{matrix}[\Box D_1\dots\Box D_m]^b\\ \mathfrak{D}\\ A\end{matrix}}{F}$$

We transform this deduction as follows:

$$\dfrac{\mathfrak{D}'_i \qquad {}_{a,b}\dfrac{\begin{matrix}[\Box A]^a\\ \mathfrak{D}_0\\ F\end{matrix} \qquad \begin{matrix}\mathfrak{D}_1\dots\mathfrak{D}_{i-1}\\ \Box D_1\dots\Box D_{i-1}\end{matrix} \qquad \begin{matrix}\mathfrak{D}''_i\\ \Box D_i\end{matrix} \qquad \begin{matrix}\mathfrak{D}_{i+1}\dots\mathfrak{D}_m\\ \Box D_{i+1}\dots\Box D_m\end{matrix} \qquad \begin{matrix}[\Box D_1\dots\Box D_m]^b\\ \mathfrak{D}\\ A\end{matrix}}{F}}{F}$$

Case 2. $\Box D_i$ is obtained as a result of an application of $(\Box GI)$.

$$\dfrac{\begin{matrix}[\Box A]^a\\ \mathfrak{D}_0\\ F\end{matrix} \quad \begin{matrix}\mathfrak{D}_1\dots\mathfrak{D}_{i-1}\\ \Box D_1\dots\Box D_{i-1}\end{matrix} \quad \dfrac{\begin{matrix}[\Box C]^c\\ \mathfrak{E}_0\\ \Box D_i\end{matrix} \quad \begin{matrix}\mathfrak{E}_1\dots\mathfrak{E}_k\\ \Box E_1\dots\Box E_k\end{matrix} \quad \begin{matrix}[\Box E_1\dots\Box E_k]^d\\ \mathfrak{E}\\ C\end{matrix}}{\Box D_i}{}_{c,d} \quad \begin{matrix}\mathfrak{D}_{i+1}\dots\mathfrak{D}_l\\ \Box D_{i+1}\dots\Box D_l\end{matrix} \quad \begin{matrix}[\Box D_1\dots\Box D_l]^b\\ \mathfrak{D}\\ A\end{matrix}}{F}{}_{a,b}$$

Then we transform the deduction as follows.

$$\dfrac{\begin{matrix}[\Box A]^a\\ \mathfrak{D}_0\\ F\end{matrix} \quad \begin{matrix}\mathfrak{D}_1\dots\mathfrak{D}_{i-1}\\ \Box D_1\dots\Box D_{i-1}\end{matrix} \begin{matrix}\mathfrak{E}_1\dots\mathfrak{E}_k\\ \Box E_1\dots\Box E_k\end{matrix} \begin{matrix}\mathfrak{D}_{i+1}\dots\mathfrak{D}_l\\ \Box D_{i+1}\dots\Box D_l\end{matrix} \quad \begin{matrix}[\Box D_1\dots\Box D_{i-1}] \quad \dfrac{\begin{matrix}[\Box C]^c\\ \mathfrak{E}_0\\ \Box D_i\end{matrix} \quad \begin{matrix}\mathfrak{E}_1\dots\mathfrak{E}_k\\ \Box E_1\dots\Box E_k\end{matrix} \quad \begin{matrix}[\Box E_1\dots\Box E_k]^d\\ \mathfrak{E}\\ C\end{matrix}}{\Box D_i}{}_{c,d} \quad [\Box D_{i+1}\dots\Box D_l]\\ \ddots\vdots\cdot^{\cdot^{\cdot}}\\ \mathfrak{D}\\ A\end{matrix}}{F}{}_{a,b}$$

The commuting conversions for **S5** are similar.

Theorem 67.

- *If $\Gamma \vdash_{\mathbf{L}} A$, where $\mathbf{L} \in \{\mathbf{S4}, \mathbf{S5}\}$, then there is a deduction in normal form with an occurrence of A as the conclusion and occurrences of the formulas in Γ as the undischarged assumptions.*

- *If $\mathfrak{D}$ is a deduction in normal form, then all major premises of elimination rules are (discharged or undischarged) assumptions of $\mathfrak{D}$.*

- *If any major premises of elimination rules are on a branch in a deduction in normal form, then they precede any major assumptions discharged by introduction rules that are on the branch.*

- *Deductions in normal forms in* **S4** *and* **S5** *have the subformula property.*

Proof. By the induction on order of branches. □

As for the natural deduction rules for non-standard modalities, we can quite easily obtain the rules for negated modalities, using their definitions $\sim A = \neg\Box A = \Diamond\neg A$ and $\dot{\sim} A = \neg\Diamond A = \Box\neg A$ (these rules are supposed to be used together with the rules for $\neg$):

$$(\sim GI)^a\ \dfrac{\begin{array}{cc} [\sim A]^a & \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ F & \neg A \end{array}}{F} \qquad (\sim E)^a\ \dfrac{\begin{array}{ccc} & & [\neg A, B_1 \dots B_m]^a \\ \mathfrak{D}_0 & \mathfrak{D}_1 \dots \mathfrak{D}_m & \mathfrak{E} \\ \sim A & B_1 \dots B_m & C \end{array}}{C}$$

$$(\dot{\sim} GI)^{a,b}\ \dfrac{\begin{array}{ccc} [\dot{\sim} A]^a & & [B_1 \dots B_m]^b \\ \mathfrak{D}_0 & \mathfrak{D}_1 \dots \mathfrak{D}_m & \mathfrak{E} \\ F & B_1 \dots B_m & \neg A \end{array}}{F} \qquad (\dot{\sim} GE)^a\ \dfrac{\begin{array}{cc} & [\neg A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \dot{\sim} A & F \end{array}}{F}$$

$B_1, \dots, B_m$ are exactly the undischarged assumptions in $\mathfrak{D}$ and $\neg A, B_1, \dots, B_m$ in $\mathfrak{E}$. For **S4**, for the case when $\sim$ is in the language, $B_1, \dots, B_m$ are required to be of the form $\neg{\sim} D_1, \dots, \neg{\sim} D_m$ (it makes these formulas equivalent to $\Box D_1, \dots, \Box D_m$) and C to be of the form $\sim D$; for the case when $\dot{\sim}$ is in the language, $B_1, \dots, B_m$ are required to be of the form $\dot{\sim} D_1, \dots, \dot{\sim} D_m$ (it makes these formulas equivalent to $\Box\neg D_1, \dots, \Box\neg D_m$) and C to be of the form $\neg\dot{\sim} D$ (it makes it equivalent to $\Diamond D$). For **S5**-style logics, $A, B_1, \dots, B_m, C$ are required to be modalized (the formulas $\sim A$ and $\dot{\sim} A$ are treated as modalized).

Theorem 68. *Let* $\mathbf{L} \in \{\mathbf{S4}^{\sim}, \mathbf{S4}^{\dot{\sim}}, \mathbf{S5}^{\sim}, \mathbf{S5}^{\dot{\sim}}\}$. *For any formula A, A is valid in* $\mathbf{L}$ *iff it is provable in the natural deduction system for* $\mathbf{L}$.

Proof. Follows from Theorem 65 and the definitions of $\sim$ and $\dot{\sim}$. □

The reduction procedures for $\sim$ and $\dot{\sim}$ are similar to the above described for $\Box$ and $\Diamond$. By the methods similar to the above applied, we can obtain the following theorem.

Theorem 69.
- *Any deduction in* $\mathbf{S4}^{\sim}$, $\mathbf{S4}^{\dot{\sim}}$, $\mathbf{S5}^{\sim}$ *and* $\mathbf{S5}^{\dot{\sim}}$ *can be converted into a deduction in normal form.*
- *Deductions in normal forms in* $\mathbf{S4}^{\sim}$, $\mathbf{S4}^{\dot{\sim}}$, $\mathbf{S5}^{\sim}$ *and* $\mathbf{S5}^{\dot{\sim}}$ *have the negation subformula property.*

As for the other modalities, the situation is more difficult. Consider the modality of non-contingency $\rhd A = \Box A \vee \Box\neg A$ and the modality of contingency $\blacktriangleright A = \Diamond A \wedge \Diamond\neg A$. Let us try to present the elimination rules for $\rhd A$. Since $\rhd A = \Box A \vee \Box\neg A$, it should be an instance of the $\vee$ elimination rule such that its subderivations are instances of the $\Box$ elimination rule. So we get something like this:

$$ {}^{c,d}\ \dfrac{\begin{array}{c} \mathfrak{D}_1 \\ \rhd A \end{array} \quad {}^{a}\dfrac{[\Box A]^c \quad \begin{array}{c} [A]^a \\ \mathfrak{D}_2 \\ B \end{array}}{B} \quad {}^{b}\dfrac{[\Box\neg A]^d \quad \begin{array}{c} [\neg A]^b \\ \mathfrak{D}_3 \\ B \end{array}}{B}}{B}$$

or like this, if we recall that $\rhd A = A \wedge \Box A$:

$$ {}^{c,d,c',d'}\ \dfrac{\begin{array}{c} \mathfrak{D}_1 \\ \rhd A \end{array} \quad {}^{a}\dfrac{[A]^c \quad [\rhd A]^{c'} \quad \begin{array}{c} [A]^a \\ \mathfrak{D}_2 \\ B \end{array}}{B} \quad {}^{b}\dfrac{[\neg A]^d \quad [\rhd\neg A]^{d'} \quad \begin{array}{c} [\neg A]^b \\ \mathfrak{D}_3 \\ B \end{array}}{B}}{B}$$

Strictly speaking, we do not have the rules yet, just some deductions, but notice that the conclusion of these deductions can be obtained just by an application of the rule (EM) without any reference to $\rhd A$:

$$\dfrac{\begin{array}{c}[A]^a\\ \mathfrak{D}_2\\ B\end{array}\quad\begin{array}{c}[\neg A]^b\\ \mathfrak{D}_3\\ B\end{array}}{F}$$

So it does not seem to be the way one can obtain the elimination rules for $\rhd$. A similar problem arises with the introduction rule for $\blacktriangleright$. Since $\blacktriangleright A = \Diamond A \wedge \Diamond \neg A$ we need to combine the $\wedge$ introduction rule with the $\Diamond$ introduction one. The result is as follows:

$$\dfrac{\begin{array}{c}[\blacktriangleright A]^a\\ \mathfrak{D}_0\\ B\end{array}\quad \dfrac{\begin{array}{c}[\Diamond A]^a\\ \mathfrak{D}_1\\ B\end{array}\quad\begin{array}{c}\mathfrak{D}_2\\ A\end{array}}{B}\quad \dfrac{\begin{array}{c}[\Diamond \neg A]^a\\ \mathfrak{D}_3\\ B\end{array}\quad\begin{array}{c}\mathfrak{D}_4\\ \neg A\end{array}}{B}}{B}$$

As we can see, it is just an instance of (EFQ):

$$\dfrac{\begin{array}{c}\mathfrak{D}_2\\ A\end{array}\quad\begin{array}{c}\mathfrak{D}_4\\ \neg A\end{array}}{B}$$

Unfortunately, at the moment we cannot offer any solution to these problems and leave the task of providing adequate natural deduction rules for $\rhd$ and $\blacktriangleright$ (and other non-standard modalities, except $\dot{\sim}$ and $\sim$) for future research. However, this situation can teach us some things. First of all, despite the fact that non-standard modalities are expressed via standard ones (and in reflexive/serial logics vice versa), the task of the providing proof theory for them does not seem to be obvious, and one may run into unexpected problems. Second, hypersequent and nested sequent calculi frameworks seem to be more convenient and suitable for studying these modalities than ordinary sequent calculi (recall Zolin's [200, 201] problems with cut elimination for sequent calculi for non-contingency logics, even for the logics that enjoy cut elimination in the language with $\Box$ or $\Diamond$) and natural deduction systems (recall our problems). Third, it might be the case that in order to provide a natural deduction formulation of non-standard modalities, one might need some generalisation of a natural deduction system similar to the generalisation of ordinary sequent calculi by hypersequent and nested sequent calculi. This could be a starting point for fruitful further research in the area of natural deduction systems.

Chapter 3

Proof systems for selected many-valued logics

3.1 Preface

Correspondence analysis is a uniform method of constructing proof systems for many-valued logics.[1] One of its applications is a general production of natural deduction systems for three-valued logics: having negation as a basic connective, one may obtain in one go rules for any n-ary three-valued connective. The aim of this section is to uniformly prove normalisation for all tabular n-ary extensions of the negation fragments of three-valued logics with Łukasiewicz's, Heyting's, and Bochvar's negations.

The roots of the method of correspondence analysis go back to Segerberg's paper [173]. This method is based on the 1-1 correspondence between inference rules and truth table entries. It generates mainly natural deduction systems (but not only them; see [112, 113, 97] where it was used to produce sequent calculi and the so-called calculus of Socratic proofs) in a uniform way for a plenty of truth-functional finitely-valued logics per saltum. Segerberg himself presented natural deduction systems for all Boolean n-ary connectives. Several decades later, independently from Segerberg the same idea was rediscovered by Kooi and Tamminga [96] (the term correspondence analysis is theirs): they introduced a natural deduction system for the logic of paradox **LP** [1, 162] itself and, using the above-mentioned 1-1 correspondence between rules and entries, formulated natural deduction systems for all truth-functional extensions of **LP** by unary and binary logical connectives. After that, Tamminga [189] obtained a similar result for strong Kleene logic $\mathbf{K_3}$ [95]. This 1-1 correspondence was supplemented in [151] where in some cases two rules of an inference correspond to one truth-table entry: it allowed to spread the method for the case of Belnap-Dunn's [13, 14, 34] four-valued logic **FDE** in [151] and generalise Kooi and Tamminga's [96, 189] results in [145] for a wider class of three-valued logics, including Heyting-Gödel's logic $\mathbf{G_3}$ [74, 67] and its dual $\mathbf{DG_3}$ [142]. Correspondence analysis was also used for formalisations of four-valued logics of rational agents [146], for syntactical investigation [153] of Tomova's natural logics [190], for the study of proof-theoretic, functional, and erotetic (i.e., pertaining to the logic of questions) aspects of Boolean binary connectives [112, 113, 152]. The papers [149, 150, 151] contain automated proof-searching procedures for some of the calculi obtained by correspondence analysis. Finally, let us mention two most recent papers on correspondence analysis: one was written by Kooi and Tamminga [97] and is devoted to the formalisation of Belnap-Dunn-style logics via sequent calculi (in contrast to [151] where natural deduction systems were used); another by Petrukhin and Shangin [154] which is devoted to natural

[1] This Section as well as Sections 3.2, 3.3, and 3.4 are based on the results of the unpublished joint paper by the author and Nils Kürbis [106]. Sections 3.1, 3.2 and 3.3 are entirely written by the author; in Section 3.4, Lemmas 98 and 99 are proven by N. Kürbis, the formulations of Definitions 95, 96, and 97 are due to N. Kürbis; as for the rest of Section 3.4, the author and N. Kürbis have contributed equally.

deduction systems for a class of three-valued logics with the non-transitive consequence relation in the style of Weir [196].

However, none of the above-mentioned papers considers the problem of normalisation. Since correspondence analysis usually produces natural deduction systems and one of the most crucial theorems regarding properties of this type of calculus is normalisation, it is important for the further development of correspondence analysis to show that natural deduction systems produced by this method enjoy normalisation.

One of the important consequences of the normalisation theorem (and shape of the rules) is the *subformula property*: a deduction has it iff "any formula that occurs on it is a subformula of either an undischarged assumption or of the conclusion." [104, p. 14224] In the case of our logics, because of the shape of the rules we are not able to obtain the subformula property, but we can still reach the *negation* subformula property: formulas in a deduction are subformulas or negations of subformulas of either an undischarged assumption or of the conclusion. Despite the fact that we need to consider special negation versions of the subformula properties, our general elimination and general introduction rules are still able to capture Dummett's [36] notions of *harmony* and *stability* which are important for a proof-theoretic semantics that defines the meaning of connectives via the rules for them.

Reading the papers [96, 189, 151], one probably would be rather sceptical about the possibility of normalised correspondence analysis, because quite often this method produces the rules of the shapes (where $\circ$ is some binary connective) $B \wedge \neg B \vdash ((A \circ B) \wedge \neg(A \circ B)) \vee \neg A$ or $\neg B, (A \circ B) \vee \neg(A \circ B) \vdash A \vee \neg A$, and so on. However, Segerberg's original rules as well as the rules for three-valued logics from [145] have a better shape. All of Segerberg's rules and most of the rules from [145] are either general elimination or general introduction rules in the terminology of Negri and von Plato [131]. And as we already know, Segerberg's system enjoys normalisation. We take the results from [145] as a starting point: we change some of the rules a bit (to make all of them general elimination or general introduction ones) and present them in a uniform way for the case of n-ary connectives (in [145], only unary and binary ones are considered). Then we prove normalisation for the resulting natural deduction systems.

The possibility of proving normalisation for Segerberg's systems is mentioned in [170] and is practically established in the previous chapter of this work. In [38] his method is extended to the case of many-valued logics with the help of labelled or marked formulas, and for the resulting systems, normalisation is proved. However, it is not shown how these calculi and normalisation for them can be adapted to the case of non-labelled calculi. Yet another method of constructing natural deduction systems with the help of labelled formulas is due to Baaz, Fermüller, and Zach [10, 11]. Our natural deduction systems are purely syntactic and do not use any labelled formulas. This is an important and principal difference between our systems and the above-mentioned ones. We should also say that the idea of 1-1 correspondence between truth table entries and inference rules is used at one of the stages of Avron, Ben-Naim, and Konikowska's [8] technique of constructing sequent caluli for many-valued logics. One may find the comparison of their strategy with correspondence analysis in [151].

The results which we present in this section may be viewed as a generalisation of the completeness results from [145] and as a continuation of our previous ones, presented in [105], where we proved normalisation for several three- and four-valued logics, including $\mathbf{LP}$, $\mathbf{K_3}$, and some of their implicative extensions. However, in contrast to the current research, in [105] we considered neither a uniform approach to constructing natural deduction systems nor a uniform approach to proving normalisation for such systems. On the other hand, our results are an adaptation for the case of three-valued logics of the techniques used in [104] for a formalisation of intuitionistic logic with general elimination and general introduction rules.

3.2 Correspondence analysis for three-valued logics

3.2.1 Semantics of the logics in question

We study three-valued logics and deal with the following truth values: 1 (true), $1/2$ (intermediate value[2]), and 0 (false). In one half of the logics that we consider (**LP**, $\mathbf{DG_3}$, their negation fragments, and the extensions of their fragments by n-ary connectives), $1/2$ is designated; in another half ($\mathbf{K_3}$, $\mathbf{G_3}$, their negation fragments, and the extensions of their fragments by n-ary connectives), it is not designated.

The logic of paradox **LP** was first formulated by Asenjo [1] and later actively studied, popularised, and dubbed **LP** by Priest [162, 163]. Strong Kleene logic $\mathbf{K_3}$ arose in Kleene's paper [95], although its $\{\neg, \wedge, \vee\}$-fragment had earlier been introduced as the fragment of Łukasiewicz's logic [115]. Heyting's logic $\mathbf{G_3}$ had been first introduced by Heyting [74], but later on it was rediscovered by Gödel [67] (the name $\mathbf{G_3}$ is in his honor) and Jaśkowski [86]. Dual Heyting's logic $\mathbf{DG_3}$ was investigated by Osorio and Carballido [142] (they called it $\mathbf{G_3'}$).

We consider two propositional languages, $\mathscr{L}^{\neg}$ and $\mathscr{L}^{\neg}_{(\circledcirc)_m}$. $\mathscr{L}^{\neg}$ has the alphabet of $\langle \mathcal{P}, \neg \rangle$, where $\mathcal{P}$ is the set of propositional variables $\{p, p_0, p_1, \ldots\}$; so it is the language of the negation fragments of the logics in question[3]. $\mathscr{L}^{\neg}_{(\circledcirc)_m}$ has the alphabet $\langle \mathcal{P}, \neg, \circledcirc_1, \ldots, \circledcirc_m \rangle$, where $\mathcal{P}$ is the set of propositional variables $\{p, q, r, s, p_1, \ldots\}$, $m \geqslant 1$, $\circledcirc_i$ $(1 \leqslant i \leqslant m)$ is an n-ary connective for $n \geqslant 1$. The sets $\mathscr{F}^{\neg}$ and $\mathscr{F}^{\neg}_{(\circledcirc)_m}$ of all $\mathscr{L}^{\neg}$- and $\mathscr{L}^{\neg}_{(\circledcirc)_m}$-formulas, respectively, are defined in a standard way. Let us consider the following matrices:

A	$\neg_L$	$\neg_H$	$\neg_B$
1	0	0	0
$1/2$	$1/2$	0	1
0	1	1	1

We see here three most popular and natural[4] three-valued negations: $\neg_L$ by Łukasiewicz [115] (also studied by Kleene [95] and Bochvar [17]), $\neg_H$ by Heyting [74], and $\neg_B$ by Bochvar [17]. **LP** and $\mathbf{K_3}$ have $\neg_L$, $\mathbf{G_3}$ has $\neg_H$, and $\mathbf{DG_3}$ has $\neg_B$. In what follows, we omit the subscripts in their symbols since, from the context, it will be clear which logic and hence which negation we mean. In the further exposition, we will use the following terminology: 1-logics and 2-logics. The former logics have one designated value ($\mathbf{K_3}$ and $\mathbf{G_3}$, their negation fragments and their extensions by n-ary connectives); the latter logics have two designated values (**LP** and $\mathbf{DG_3}$, their negation fragments and their extensions by n-ary connectives). The entailment relation in these logics is defined as follows (Γ stands for a finite set of formulas and A is a formula; a valuation is understood as a mapping from the set $\mathcal{P}$ to $\{1, 1/2, 0\}$ and is extended to the case of complex formulas via truth tables (matrices)):

- if **L** is an 1-logic, then $\Gamma \models_{\mathbf{L}} A$ iff for any valuation v, if $v(B) = 1$ for any $B \in \Gamma$, then $v(A) = 1$;
- if **L** is a 2-logic, then $\Gamma \models_{\mathbf{L}} A$ iff for any valuation v, if $v(B) \neq 0$ for any $B \in \Gamma$, then $v(A) \neq 0$.

[2] It can be interpreted in various ways, e.g., paradoxical or both true and false (if it is designated), neither true nor false (if it is not designated), undefined or unknown.

[3] These logics, **LP**, $\mathbf{K_3}$, $\mathbf{G_3}$, and $\mathbf{DG_3}$ are usually considered in the $\{\neg, \vee, \wedge, \rightarrow\}$-language, although **LP** and $\mathbf{K_3}$ are quite often (and, in particular, in Kooi and Tamminga's works [96, 189]) studied in the implicationless languages, since their implication defined as $\neg A \vee B$ does not validate modus ponens, see [105] for a bit more detailed discussion.

[4] There are at least two more possibilities for defining negation in the three-valued setting: Post's [159] negation and its reverse [147]. But these two connectives belong to the class of circular negations, which 'negationness' may be debatable, although it is not an obstacle for the consideration of natural deduction, as we will see in the later sections.

An interesting feature of all the 2-logics in question is that all of them are paraconsistent[5], while all 1-logic are paracomplete[6].

We could present here a list of possible extensions of the negation fragments of **LP**, $\mathbf{K_3}$, $\mathbf{G_3}$, and $\mathbf{DG_3}$ by various connectives (and describe the other connectives of these logics); however, an extensive survey of such connectives is presented in [145], so we just refer to it and move on to the consideration of natural deduction systems.

3.2.2 The method of correspondence analysis and natural deduction systems obtained by it

Consider the subsequent rules:

$$(\mathrm{EM})^{a,b}\ \frac{\begin{matrix}[A]^a & [\neg A]^b\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ B & B\end{matrix}}{B} \qquad (\mathrm{EM}_\neg)^{a,b}\ \frac{\begin{matrix}[\neg A]^a & [\neg\neg A]^b\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ B & B\end{matrix}}{B}$$

$$(\neg\neg I)^a\ \frac{\begin{matrix} & [\neg\neg A]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ A & B\end{matrix}}{B} \qquad (\neg\neg E)^a\ \frac{\begin{matrix} & [A]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ \neg\neg A & B\end{matrix}}{B}$$

$$(\mathrm{EFQ})\ \frac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ \neg A & A\end{matrix}}{B} \qquad (\mathrm{EFQ}_\neg)\ \frac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2\\ \neg\neg A & \neg A\end{matrix}}{B}$$

- The natural deduction system $\mathbb{ND}^{\neg}_{\mathbf{LP}}$ for the negative fragment of **LP** has the rules (EM), $(\neg\neg I)$, and $(\neg\neg E)$.
- The natural deduction system $\mathbb{ND}^{\neg}_{\mathbf{DG}}$ for the negative fragment of $\mathbf{DG_3}$ has the rules (EM) and $(\mathrm{EFQ}_\neg)$.
- The natural deduction system $\mathbb{ND}^{\neg}_{\mathbf{K}}$ for the negative fragment of $\mathbf{K_3}$ has the rules $(\neg\neg I)$, $(\neg\neg E)$, and (EFQ).
- The natural deduction system $\mathbb{ND}^{\neg}_{\mathbf{G}}$ for the negative fragment of $\mathbf{G_3}$ has the rules (EFQ) and $(\mathrm{EM}_\neg)$.

Definition 70 (Deduction in $\mathbb{ND}^{\neg}_{\mathbf{L}}$)**.** Let $\mathbf{L} \in \{\mathbf{LP}, \mathbf{K}, \mathbf{G}, \mathbf{DG}\}$.

1. The formula occurrence A is a deduction in $\mathbb{ND}^{\neg}_{\mathbf{L}}$ of A from the undischarged assumption A.
2. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are deductions in $\mathbb{ND}^{\neg}_{\mathbf{L}}$, then the applications of the above-mentioned rules are deductions of B in $\mathbb{ND}^{\neg}_{\mathbf{L}}$ from the undischarged assumptions in $\mathfrak{D}_1$ and $\mathfrak{D}_2$ apart from those in the assumption classes a and b, which are discharged.
3. Nothing else is a deduction in $\mathbb{ND}^{\neg}_{\mathbf{L}}$.

We write $\Gamma \vdash_{\mathbb{ND}^{\neg}_{\mathbf{L}}} A$ (or just $\Gamma \vdash_{\mathbf{L}} A$) if there is a deduction in $\mathbb{ND}^{\neg}_{\mathbf{L}}$ of (the formula occurrence) A from (occurrences of) some of the formulas in Γ.

---

[5] At least in Priest's [163] sense: if for some formulas A and B, $A, \neg A \not\models_{\mathbf{L}} B$, then **L** is a paraconsistent logic.

[6] We understand a paracomplete logic as the one in which for some formulas A and B, $A \models_{\mathbf{L}} B$, $\neg A \models_{\mathbf{L}} B$, while $\not\models_{\mathbf{L}} B$. If we would consider a multiple conclusion entailment relation, we could use a simplier definition by Hyde [76]: $B \not\models_{\mathbf{L}} A, \neg A$. See [145] for some other definitions.

These systems were investigated in [145], but the double negation rules are not in the general elimination or introduction form. One may easily derive standard, not general, versions of the double negation rules from these ones. Now let us adopt for our case some Segerberg's notation [173].

Notation 71. Consider the set of natural numbers $\mathfrak{s} = \{1, \ldots, n\}$. By a partitioning of $\mathfrak{s}$ we mean an ordered triple $\langle I, J, K\rangle$ such that $I \cup J \cup K = \mathfrak{s}$ and $I \cap J \cap K = \emptyset$. In what follows, we are going to consider a partitioning of the following type: $\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle$, where $\mathfrak{t} = \{i \in \mathfrak{s} \mid v(A_i) = 1, A_i \in \mathscr{F}_{(\circledcirc)_m}\}$, $\mathfrak{h} = \{j \in \mathfrak{s} \mid v(A_j) = 1/2, A_j \in \mathscr{F}_{(\circledcirc)_m}\}$, and $\mathfrak{f} = \{k \in \mathfrak{s} \mid v(A_k) = 0, A_k \in \mathscr{F}_{(\circledcirc)_m}\}$.

And a few more notation, this time in the spirit of Kooi and Tamminga [96, 189].

Notation 72. The expression $f_{\circledcirc}(x_1, \ldots, x_n) = y$, where $x_1, \ldots, x_n, y \in \{1, 1/2, 0\}$, means that if $v(A_1) = x_1, \ldots, v(A_n) = x_n$, then $v(\circledcirc(A_1, \ldots, A_n)) = y$, for each valuation v and all formulas $A_1, \ldots, A_n$. The expression $f_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle\big) = x$, where $x \in \{1, 1/2, 0\}$, means that if $v(A_i) = 1$ (for each $i \in \mathfrak{t}$), $v(A_j) = 1/2$ (for each $j \in \mathfrak{h}$), and $v(A_k) = 0$ (for each $k \in \mathfrak{f}$), then $v(\circledcirc(A_1, \ldots, A_n)) = x$, for each valuation v.

Let us clarify a bit this notation and, with its help, the idea of correspondence analysis. Consider, for example, some ternary connective $\odot$, a particular instance of our $\circledcirc$, such that for any valuation v, $v(\odot(A, B, C)) = 1$, if $v(A) = 1/2$, $v(B) = 0$, and $v(C) = 1/2$. We have $f_{\odot}(1/2, 0, 1/2) = 1$ and this equality is said to be an entry. We enumerate all the formulas A, B, C as follows: A_1, A_2, A_3. We can form a triple $\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle$ such that $\langle \emptyset, \{1, 3\}, \{2\}\rangle$. Thus, we have $f_{\odot}\big(\langle \emptyset, \{1, 3\}, \{2\}\rangle\big) = 1$. By looking at the number of formulas and the structure of the triple, we can understand which values these formulas have. Such notation will help us present in a uniform way the rules that correspond to the entry. In particular, we get a rule called $R_{\odot}\big(\langle \emptyset, \{1, 3\}, \{2\}\rangle 1\big)$ (for 2-logics) and a rule called $\widetilde{R}_{\odot}\big(\langle \emptyset, \{1, 3\}, \{2\}\rangle 1\big)$ (for 1-logics). Let us present them:

$$
R_{\odot}\big(\langle \emptyset, \{1, 3\}, \{2\}\rangle 1\big)^a \;
\frac{\begin{matrix} \mathfrak{D}_1 \\ \neg\odot(A, B, C) \end{matrix} \quad \begin{matrix} \mathfrak{D}_2 \\ A \end{matrix} \quad \begin{matrix} \mathfrak{D}_3 \\ \neg A \end{matrix} \quad \begin{matrix} \mathfrak{D}_4 \\ C \end{matrix} \quad \begin{matrix} \mathfrak{D}_5 \\ \neg C \end{matrix} \quad \begin{matrix} [B]^a \\ \mathfrak{D}_6 \\ D \end{matrix}}{D}
$$

$$
\widetilde{R}_{\odot}\big(\langle \emptyset, \{1, 3\}, \{2\}\rangle 1\big)^{a,b,c,d,e} \;
\frac{\begin{matrix} [\neg\odot(A, B, C)]^a \\ \mathfrak{D}_1 \\ D \end{matrix} \quad \begin{matrix} [A]^b \\ \mathfrak{D}_2 \\ D \end{matrix} \quad \begin{matrix} [\neg A]^c \\ \mathfrak{D}_3 \\ D \end{matrix} \quad \begin{matrix} [C]^d \\ \mathfrak{D}_4 \\ D \end{matrix} \quad \begin{matrix} [\neg C]^e \\ \mathfrak{D}_5 \\ D \end{matrix} \quad \begin{matrix} \mathfrak{D}_6 \\ B \end{matrix}}{D}
$$

As we will see soon, one can prove that $f_{\odot}(1/2, 0, 1/2) = 1$ iff $R_{\odot}\big(\langle \emptyset, \{1, 3\}, \{2\}\rangle 1\big)$ is valid (for 2-logics); and $f_{\odot}(1/2, 0, 1/2) = 1$ iff $\widetilde{R}_{\odot}\big(\langle \emptyset, \{1, 3\}, \{2\}\rangle 1\big)$ is valid (for 1-logics). It is called the single entry correspondence. By the way, it is time to give the precise definition of this notion; we even give it in a generalised form, following [145, Definition 4.1]; in the original formulation, [96, Definition 2.1] and [189, Definition 1], a single entry could be characterised by a single rule only; also, this rule cannot have subdeductions (in our case, it is allowed).

Definition 73 (Generalized single entry correspondence)**.** Let $x_1, \ldots, x_n, y \in \{1, 1/2, 0\}$ and $A_1, \ldots, A_h$, $B_1, \ldots, B_g, C_1, \ldots, C_t \in \mathscr{F}_{\circledcirc(m)}$. Let E be a truth table (or matrix) entry of the type $f_{\circledcirc}(x_1, \ldots, x_n) = y$. Let $1 \leqslant l \leqslant h$ and I_l/A_l be an inference scheme of the type $B_1, \ldots, B_g/A_l$ or $B_1, \ldots, B_g, C_1 \vdash A_l, \ldots, C_t \vdash A_l/A_l$ or $C_1 \vdash A_l, \ldots, C_t \vdash A_l/A_l$. Then E is *characterised* by inference schemes $I_1/A_1, \ldots, I_h/A_h$, if

$$E \text{ if and only if } I_1 \models A_1, \ldots, I_k \models A_k.$$

Notice that all the entries of the forms $f_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle\big) = 1$ and $f_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle\big) = 0$ are characterised by one rule, while all the entries of the form $f_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle\big) = 1/2$ are characterised by two rules. So if our $\odot$ behaves in such a way that, e.g., $f_\odot(\langle\{3\},\{1\},\{2\}\rangle) = 1/2$, that is $v(\odot(A_1, A_2, A_3)) = 1/2$, if $v(A_1) = 1/2$, $v(A_2) = 0$, and $v(A_3) = 1$, then it is characterised by two rules: $R_\odot(\langle\{3\},\{1\},\{2\}\rangle 1/2)$ and $R^{\neg}_\odot(\langle\{3\},\{1\},\{2\}\rangle 1/2)$ (the case of 2-logics) or $\widetilde{R}_\odot(\langle\{3\},\{1\},\{2\}\rangle 1/2)$ and $\widetilde{R}^{\neg}_\odot(\langle\{3\},\{1\},\{2\}\rangle 1/2)$ (the case of 1-logics). Let us enlist these rules as well:

$$R_\odot\big(\langle\{3\},\{1\},\{2\}\rangle 1/2\big)^{a,b,c}\ \dfrac{\begin{array}{ccccc} [\odot(A,B,C)]^a & [\neg C]^b & & & [B]^c \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 & \mathfrak{D}_4 & \mathfrak{D}_5 \\ D & D & A & \neg A & D \end{array}}{D}$$

$$R^{\neg}_\odot\big(\langle\{3\},\{1\},\{2\}\rangle 1/2\big)^{a,b,c}\ \dfrac{\begin{array}{ccccc} [\neg\odot(A,B,C)]^a & [\neg C]^b & & & [B]^c \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 & \mathfrak{D}_4 & \mathfrak{D}_5 \\ D & D & A & \neg A & D \end{array}}{D}$$

$$\widetilde{R}_\odot\big(\langle\{3\},\{1\},\{2\}\rangle 1/2\big)^{a,b}\ \dfrac{\begin{array}{ccccc} & & [A]^a & [\neg A]^b & \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 & \mathfrak{D}_4 & \mathfrak{D}_5 \\ \odot(A,B,C) & \neg C & D & D & B \end{array}}{D}$$

$$\widetilde{R}^{\neg}_\odot\big(\langle\{3\},\{1\},\{2\}\rangle 1/2\big)^{a,b}\ \dfrac{\begin{array}{ccccc} & & [A]^a & [\neg A]^b & \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 & \mathfrak{D}_4 & \mathfrak{D}_5 \\ \neg\odot(A,B,C) & \neg C & D & D & B \end{array}}{D}$$

In order to characterise the formula $\odot(A,B,C)$ we need to consider all possible valuations of the formulas A, B, C; we get 3 values for 3 formulas, that is 27 combinations, and for each of them we have one rule or two. Adding all these rules for the natural deduction system for the negation fragment of one of the four basic logics in question, we get a sound and complete natural deduction system for an extension of the chosen negation fragment of the chosen logic by $\odot$. Surely, after that, we can add, in a similar way, other connectives for this system. Of course, we have a lot of rules. But for the case of binary and unary connectives (which are the most popular in logic), we have fewer combinations of valuations: 9 and 3, respectively. Thus, from 9 to 18 rules for a binary connective (depending on how many times it takes the value $1/2$) and from 3 to 6 rules for a unary connective. Still, that is quite a lot of rules. But that is the price for the generality of the method, which we have to pay. Instead of finding the rules in a handmade way for any logic individually, we get the rules for all the logics in one go.

Now we are ready to present in a uniform way rules for an n-ary connective $\circledcirc$.

- The rules for **LP** and **DG$_3$**:

$$R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)^{a,b}\ \dfrac{\begin{array}{ccccc} & [\neg A_i^\dagger]^a & & & [A_k^*]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2^\dagger & \mathfrak{D}_3^\ddagger & \mathfrak{D}_4^\ddagger & \mathfrak{D}_5^* \\ \neg\circledcirc(A_1,\ldots,A_n) & B & A_j^\ddagger & \neg A_j^\ddagger & B \end{array}}{B}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

$$R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)^{a,b,c}\ \dfrac{\begin{array}{ccccc} [\circledcirc(A_1,\ldots,A_n)]^a & [\neg A_i^\dagger]^b & & & [A_k^*]^c \\ \mathfrak{D}_1 & \mathfrak{D}_2^\dagger & \mathfrak{D}_3^\ddagger & \mathfrak{D}_4^\ddagger & \mathfrak{D}_5^* \\ B & B & A_j^\ddagger & \neg A_j^\ddagger & B \end{array}}{B}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

$$R^{\neg}_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle{}^1\!/_2\big)^{a,b,c}\ \frac{\begin{array}{ccccc} [\neg\circledcirc(A_1,\ldots,A_n)]^a & [\neg A_i^\dagger]^b & & & [A_k^*]^c \\ \mathfrak{D}_1 & \mathfrak{D}_2^\dagger & \mathfrak{D}_3^\ddagger & \mathfrak{D}_4^\ddagger & \mathfrak{D}_5^* \\ B & B & A_j^\ddagger & \neg A_j^\ddagger & B\end{array}}{B}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

$$R_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)^{a,b}\ \frac{\begin{array}{ccccc} & [\neg A_i^\dagger]^a & & & [A_k^*]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2^\dagger & \mathfrak{D}_3^\ddagger & \mathfrak{D}_4^\ddagger & \mathfrak{D}_5^* \\ \circledcirc(A_1,\ldots,A_n) & B & A_j^\ddagger & \neg A_j^\ddagger & B\end{array}}{B}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

- The rules for $\mathbf{K_3}$ and $\mathbf{G_3}$:

$$\widetilde{R}_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)^{a,b,c}\ \frac{\begin{array}{ccccc} [\circledcirc(A_1,\ldots,A_n)]^a & & [A_j^\ddagger]^b & [\neg A_j^\ddagger]^c & \\ \mathfrak{D}_1 & \mathfrak{D}_2^\dagger & \mathfrak{D}_3^\ddagger & \mathfrak{D}_4^\ddagger & \mathfrak{D}_5^* \\ B & A_i^\dagger & B & B & \neg A_k^*\end{array}}{B}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

$$\widetilde{R}_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle{}^1\!/_2\big)^{a,b}\ \frac{\begin{array}{ccccc} & & [A_j^\ddagger]^a & [\neg A_j^\ddagger]^b & \\ \mathfrak{D}_1 & \mathfrak{D}_2^\dagger & \mathfrak{D}_3^\ddagger & \mathfrak{D}_4^\ddagger & \mathfrak{D}_5^* \\ \circledcirc(A_1,\ldots,A_n) & A_i^\dagger & B & B & \neg A_k^*\end{array}}{B}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

$$\widetilde{R}^{\neg}_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle{}^1\!/_2\big)^{a,b}\ \frac{\begin{array}{ccccc} & & [A_j^\ddagger]^a & [\neg A_j^\ddagger]^b & \\ \mathfrak{D}_1 & \mathfrak{D}_2^\dagger & \mathfrak{D}_3^\ddagger & \mathfrak{D}_4^\ddagger & \mathfrak{D}_5^* \\ \neg\circledcirc(A_1,\ldots,A_n) & A_i^\dagger & B & B & \neg A_k^*\end{array}}{B}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

$$\widetilde{R}_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)^{a,b,c}\ \frac{\begin{array}{ccccc} [\neg\circledcirc(A_1,\ldots,A_n)]^a & & [A_j^\ddagger]^b & [\neg A_j^\ddagger]^c & \\ \mathfrak{D}_1 & \mathfrak{D}_2^\dagger & \mathfrak{D}_3^\ddagger & \mathfrak{D}_4^\ddagger & \mathfrak{D}_5^* \\ B & A_i^\dagger & B & B & \neg A_k^*\end{array}}{B}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

By $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$, $\mathbf{DG}_{\mathbf{3}\neg}^{(\circledcirc)_m}$, $\mathbf{K}_{\mathbf{3}\neg}^{(\circledcirc)_m}$, and $\mathbf{G}_{\mathbf{3}\neg}^{(\circledcirc)_m}$ we denote extensions of $\mathbf{LP}_{\neg}$, $\mathbf{DG}_{\mathbf{3}\neg}$, $\mathbf{K}_{\mathbf{3}\neg}$, and $\mathbf{G}_{\mathbf{3}\neg}$ by $\circledcirc_1, \ldots, \circledcirc_m$. Let $\mathbf{L} \in \{\mathbf{LP}, \mathbf{DG}, \mathbf{K}, \mathbf{G}\}$. Then $\mathbb{ND}_{\mathbf{L}}^{\neg\circledcirc}$ is a natural deduction system for $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$ (resp. $\mathbf{DG}_{\mathbf{3}\neg}^{(\circledcirc)_m}$, $\mathbf{K}_{\mathbf{3}\neg}^{(\circledcirc)_m}$, $\mathbf{G}_{\mathbf{3}\neg}^{(\circledcirc)_m}$).

Definition 74 (Deduction in $\mathbb{ND}_{\mathbf{L}}^{\neg\circledcirc}$)**.**

1. The formula occurrence A is a deduction in $\mathbb{ND}_{\mathbf{L}}^{\neg\circledcirc}$ of A from the undischarged assumption A.

2. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are deductions in $\mathbb{ND}_{\mathbf{L}}^{\neg\circledcirc}$, then the applications of the above-mentioned rules for negations are deductions of B in $\mathbb{ND}_{\mathbf{L}}^{\neg}$ from the undischarged assumptions in $\mathfrak{D}_1$ and $\mathfrak{D}_2$ apart from those in the assumption classes a and b, which are discharged.

3. If $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\mathfrak{D}_4$, and $\mathfrak{D}_5$ are deductions in $\mathbb{ND}^{\neg\circledcirc}_{\mathbf{L}}$, then the applications of the above-mentioned rules for $\circledcirc$ are deductions of B in $\mathbb{ND}^{\neg\circledcirc}_{\mathbf{L}}$ from the undischarged assumptions in $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\mathfrak{D}_4$, and $\mathfrak{D}_5$ apart from those in the assumption classes a, b, and c, which are discharged.

4. Nothing else is a deduction in $\mathbb{ND}^{\neg\circledcirc}_{\mathbf{L}}$.

We write $\Gamma \vdash_{\mathbb{ND}^{\neg\circledcirc}_{\mathbf{L}}} A$ (or just $\Gamma \vdash_{\mathbf{L}} A$) if there is a deduction in $\mathbb{ND}^{\neg\circledcirc}_{\mathbf{L}}$ of (the formula occurrence) A from (occurrences of) some of the formulas in Γ.

Let us present an adaptation of [104, Definition 3] for our case.

Definition 75 (Terminology for Premises and Discharged Assumptions)**.**

1. In applications of the general elimination rules, formula occurrences taking the places of $\neg\neg A$ (rules $(\neg\neg E)$ and (EFQ$_\neg$)), $\neg A$ (the rule (EFQ)), $\circledcirc(\vec{A})$ (rules $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ and $\widetilde{R}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$), and $\neg\circledcirc(\vec{A})$ (rules $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$ and $\widetilde{R}^{\neg}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$) to the very left above the line are the *major premises*; formula occurrences taking the places of B at the end of subdeductions are the *arbitrary premises*; formula occurrences taking the place of A in an application of (EFQ), $\neg A$ in an application of (EFQ$_\neg$), $A_j, \neg A_j$ in applications of $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ and $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$, $A_i, \neg A_k$ in applications of $\widetilde{R}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ and $\widetilde{R}^{\neg}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ are the *minor premises*.

2. In applications of the general introduction rules, formula occurrences taking the place of A in an application of $(\neg\neg I)$, $A_j, \neg A_j$ in applications of $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ and $R^{\neg}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$, $A_i, \neg A_k$ in applications of $\widetilde{R}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$ and $\widetilde{R}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ are the *specific premises*; formula occurrences taking the places of B at the end of subdeductions are the *arbitrary premises*; formula occurrences taking the places of the discharged assumptions $\neg A$ (the rule (EM)), $\neg\neg A$ (rules (EM$_\neg$), $(\neg\neg I)$), $\circledcirc(\vec{A})$ (rules $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ and $\widetilde{R}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$), and $\neg\circledcirc(\vec{A})$ (rules $\widetilde{R}^{\neg}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ and $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$) are *the major assumptions* discharged by applications of the respective rules; formula occurrences taking the place of the discharged assumptions A in (EM), $\neg A$ in (EM$_\neg$), $\neg A_i$ and A_k in $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ and $R^{\neg}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$, A_j and $\neg A_j$ in $\widetilde{R}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$ and $\widetilde{R}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ are the *minor assumptions* discharged by applications of the respective rules.

Theorem 76. *Let* $\mathbf{L}$ *be* $\mathbf{LP}^{(\circledcirc)_m}_{\neg}$ *or* $\mathbf{DG}^{(\circledcirc)_m}_{\mathbf{3}\neg}$. *Then*:

(1) $f_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle\big) = 0$ *iff* $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ *is sound in* $\mathbf{L}$.

(2) $f_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle\big) = 1/2$ *iff both* $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ *and* $R^{\neg}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ *are sound in* $\mathbf{L}$.

(3) $f_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle\big) = 1$ *iff* $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$ *is sound in* $\mathbf{L}$.

Proof. (1) By contraposition. Suppose that $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ is not sound in $\mathbf{L}$. Then there is a valuation v such that $v(\circledcirc(A_1,\ldots,A_n)) \neq 0$, $v(\neg A_i) \neq 0$, for each $i \in \mathfrak{t}$, implies $v(B) \neq 0$, $v(A_j) \neq 0$ and $v(\neg A_j) \neq 0$, for each $j \in \mathfrak{h}$, $v(A_k) \neq 0$, for each $k \in \mathfrak{f}$, implies $v(B) \neq 0$, while $v(B) = 0$. Then $v(\neg A_i) = 0$, for each $i \in \mathfrak{t}$, and $v(A_k) = 0$, for each $k \in \mathfrak{f}$. Hence, $v(A_i) = 1$, for each $i \in \mathfrak{t}$, and $v(A_j) = 1/2$, for each $j \in \mathfrak{h}$. But then $f_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle\big) \neq 0$.

Suppose that $f_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle\big) \neq 0$. Then there is a valuation v such that for some formulas $A_1,\ldots,A_n$, it holds that $v(\circledcirc(A_1,\ldots,A_n)) \neq 0$ and $A_1,\ldots,A_n$ have the $\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle$-appropriate values. Let B be such a formula that $v(B) = 0$. Then, since $v(B) = 0$, we get the following implications: $v(A_k) = 0$, for each $k \in \mathfrak{f}$, implies $v(B) = 0$; $v(\neg A_i) = 0$, for each $i \in \mathfrak{t}$, implies $v(B) = 0$ (since $v(A_i) = 1$, for each $i \in \mathfrak{t}$, implies $v(\neg A_i) = 0$, for each $i \in \mathfrak{t}$). Thus, the subdeduction of B from the set of formulas $\{\alpha_k \mid k \in \mathfrak{f}\}$ and the subdeduction of B from the set of formulas $\{\neg\alpha_i \mid i \in \mathfrak{t}\}$ are semantically correct. Also, we know that $v(A_j) = 1/2$ and $v(\neg A_j) \neq 0$, for each $j \in \mathfrak{h}$. Thus, all the subdeductions of the rule $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 0)$ are semantically correct, but not its conclusion. Hence, $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 0)$ is not sound in **L**.

(2) By contraposition. Suppose $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ or $R^{\neg}_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ are not sound in **L**. Suppose $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ is not sound in **L**. Then there is a valuation v such that $v(\circledcirc(A_1, \ldots, A_n)) \neq 0$ implies $v(B) \neq 0$, $v(\neg A_i) \neq 0$, for each $i \in \mathfrak{t}$, implies $v(B) \neq 0$, $v(A_j) \neq 0$ and $v(\neg A_j) \neq 0$, for each $j \in \mathfrak{h}$, $v(A_k) \neq 0$, for each $k \in \mathfrak{f}$, implies $v(B) \neq 0$, while $v(B) = 0$. Then $v(\circledcirc(A_1, \ldots, A_n)) = 0$, $v(A_i) = 1$, for each $i \in \mathfrak{t}$, $v(A_j) = 1/2$, for each $j \in \mathfrak{h}$, and $v(A_k) = 0$, for each $k \in \mathfrak{f}$. Hence, $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) \neq 1/2$. Assume that $R^{\neg}_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ is not sound in **L**. Then there is a valuation v such that $v(\neg\circledcirc(A_1, \ldots, A_n)) \neq 0$ implies $v(B) \neq 0$, $v(\neg A_i) \neq 0$, for each $i \in \mathfrak{t}$, implies $v(B) \neq 0$, $v(A_j) \neq 0$ and $v(\neg A_j) \neq 0$, for each $j \in \mathfrak{h}$, $v(A_k) \neq 0$, for each $k \in \mathfrak{f}$, implies $v(B) \neq 0$, while $v(B) = 0$. Then $v(\circledcirc(A_1, \ldots, A_n)) = 1$, $v(A_i) = 1$, for each $i \in \mathfrak{t}$, $v(A_j) = 1/2$, for each $j \in \mathfrak{h}$, and $v(A_k) = 0$, for each $k \in \mathfrak{f}$. Hence, $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) \neq 1/2$.

Suppose $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) \neq 1/2$. Thus, there is a valuation v such that for some formulas $A_1, \ldots, A_n$, it holds that $v(\circledcirc(A_1, \ldots, A_n)) \neq 1/2$, $v(\neg A_i) = 0$, for each $i \in \mathfrak{t}$, $v(A_j) = 1/2$ and $v(\neg A_j) \neq 0$, for each $j \in \mathfrak{h}$, $v(A_k) = 0$, for each $k \in \mathfrak{f}$. Let B such a formula that $v(B) = 0$. Then, since $v(B) = 0$, we get the following implications: $v(A_k) = 0$, for each $k \in \mathfrak{f}$, implies $v(B) = 0$; $v(\neg A_i) = 0$, for each $i \in \mathfrak{t}$, implies $v(B) = 0$. Thus, the subdeduction of B from the set of formulas $\{\alpha_k \mid k \in \mathfrak{f}\}$ and the subdeduction of B from the set of formulas $\{\neg\alpha_i \mid i \in \mathfrak{t}\}$ are semantically correct. Since $v(\circledcirc(A_1, \ldots, A_n)) \neq 1/2$, either $v(\circledcirc(A_1, \ldots, A_n)) = 0$ or $v(\neg\circledcirc(A_1, \ldots, A_n)) = 0$. Since $v(B) = 0$, either $v(\circledcirc(A_1, \ldots, A_n)) = 0$ implies $v(B) = 0$, or $v(\neg\circledcirc(A_1, \ldots, A_n)) = 0$ implies $v(B) = 0$. Hence, either the subdeduction of B from $\circledcirc(A_1, \ldots, A_n)$ is semantically correct, or the subdeduction of B from $\neg\circledcirc(A_1, \ldots, A_n)$ is semantically correct. Therefore, either all the subdeductions of the rule $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ are semantically correct, but not its conclusion, or all the subdeductions of the rule $R^{\neg}_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ are semantically correct, but not its conclusion. Thus, either $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ is not sound in **L** or $R^{\neg}_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ is not sound in **L**.

(3) Similarly to (1). □

To be sure, it cannot be the case that both $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 0$ and $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 1/2$. Although, of course, $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 0$ and $f_{\circledcirc}(\langle \mathfrak{t}', \mathfrak{h}', \mathfrak{f}' \rangle) = 1/2$ can be the case for two different tuples. Thus, the rules $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1)$, $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$, and $R^{\neg}_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ cannot be in a natural deduction system at the same time, otherwise we lose soundness. However, it can be the case that $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1)$ and $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ are present in a system at the same time without lose of soundness and under the condition that $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 1$. Additionally, if $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 0$, $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 0)$ and $R^{\neg}_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ can be present in a system at the same time as well. From the point of view of completeness, in these two situations the rules dealing with $1/2$ are superfluous, but they will be extremely helpful for the normalisation theorem. So when we prove normalisation for $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$, we will presuppose that if $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1)$ is in a system, then $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ is as well (but not $R^{\neg}_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$, we do not want to lose soundness), and if $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 0)$ is in a system, then $R^{\neg}_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ is as well (but not $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$, of course).

Theorem 77. (1) *If $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 0$, then $R^{\neg}_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ is sound in $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$.*

(2) *If $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 1$, then $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ is sound in $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$.*

Proof. (1) Suppose that $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 0$ and $R^{\neg}_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ is not sound in $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$. Then there is a valuation v such that $v(\neg\circledcirc(A_1, \ldots, A_n)) \neq 0$ implies $v(B) \neq 0$, $v(\neg A_i) \neq 0$, for each $i \in \mathfrak{t}$, implies $v(B) \neq 0$, $v(A_j) \neq 0$ and $v(\neg A_j) \neq 0$, for each $j \in \mathfrak{h}$, $v(A_k) \neq 0$, for each $k \in \mathfrak{f}$, implies $v(B) \neq 0$, while $v(B) = 0$. Then $v(\circledcirc(A_1, \ldots, A_n)) = 1$, $v(\neg A_i) = 1$, $v(A_j) = 1/2$, and $v(A_k) = 0$. But then $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) \neq 0$. Contradiction. Hence, $R^{\neg}_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ is sound in $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$.

(2) Similarly to (1). □

Proposition 78. $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1/2)$ *is derivable in* $\mathbb{ND}_{\mathbf{LP}}^{\neg(\circledcirc)_m}$ *via* $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 1)$ *and* (EM).

Proof.

$$\dfrac{\begin{array}{c}[\circledcirc(\vec{A})]^1\\ \mathfrak{D}_1\\ B\end{array} \qquad \dfrac{[\neg\circledcirc(\vec{A})]^2 \quad \begin{array}{c}[\neg A_i^\dagger]^3\\ \mathfrak{D}_2\\ B\end{array} \quad \begin{array}{c}\mathfrak{D}_3\\ A_j^\ddagger\end{array} \quad \begin{array}{c}\mathfrak{D}_4\\ \neg A_j^\ddagger\end{array} \quad \begin{array}{c}[A_k^*]^4\\ \mathfrak{D}_5\\ B\end{array}}{B}\,R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)^{3,4}}{B}\,(\mathrm{EM})^{1,2}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$. □

Notice that this proof is not normal, $\neg\circledcirc(\vec{A})$ is a maximal formula here, that is why although $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ is derivable, we need to postulate it as a primitive rule to prove normalisation.

Proposition 79. *$R^\neg_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ is derivable in $\mathbb{ND}^{\neg(\circledcirc)_m}_{\mathbf{LP}}$ via $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ and* (EM)*.*

Proof.

$$\dfrac{\dfrac{[\circledcirc(\vec{A})]^1 \quad \begin{array}{c}[\neg A_i^\dagger]^2\\ \mathfrak{D}_1\\ B\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ A_j^\ddagger\end{array} \quad \begin{array}{c}\mathfrak{D}_3\\ \neg A_j^\ddagger\end{array} \quad \begin{array}{c}[A_k^*]^3\\ \mathfrak{D}_4\\ B\end{array}}{B}\,R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)^{2,3} \qquad \begin{array}{c}[\neg\circledcirc(\vec{A})]^4\\ \mathfrak{D}_5\\ B\end{array}}{B}\,(\mathrm{EM})^{1,4}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$. □

Theorem 80. *Let $\mathbf{L}$ be $\mathbf{K}^{(\circledcirc)_m}_{\mathbf{3}\neg}$ or $\mathbf{G}^{(\circledcirc)_m}_{\mathbf{3}\neg}$. Then:*

(1) *$f_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle\big) = 0$ iff $\widetilde{R}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ is sound in $\mathbf{L}$.*

(2) *$f_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle\big) = 1/2$ iff both $\widetilde{R}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ and $\widetilde{R}^\neg_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ are sound in $\mathbf{L}$.*

(3) *$f_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle\big) = 1$ iff $\widetilde{R}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$ is sound in $\mathbf{L}$.*

Proof. Similarly to Theorem 76. □

In the case of $\mathbf{K}^{(\circledcirc)_m}_{\mathbf{3}\neg}$ and $\mathbf{G}^{(\circledcirc)_m}_{\mathbf{3}\neg}$, to prove normalisation, we do not need to postulate as primitive any derivable rules.

3.3 Soundness and completeness

Lemma 81. *All the rules of $\mathfrak{ND}^\neg_{\mathbf{LP}}$, $\mathfrak{ND}^\neg_{\mathbf{DG}}$, $\mathfrak{ND}^\neg_{\mathbf{K}}$, and $\mathfrak{ND}^\neg_{\mathbf{G}}$ are sound.*

Proof. As an example, consider $\mathfrak{ND}^\neg_{\mathbf{LP}}$ and the rule (EM). Suppose that $A \models_{\mathbf{LP}} B$ and $\neg A \models_{\mathbf{LP}} B$. Then for any valuation v, it holds that if $v(A) \neq 0$, then $v(B) \neq 0$ as well as if $v(\neg A) \neq 0$, then $v(B) \neq 0$. Let $v(A) \neq 0$. Then $v(B) \neq 0$. Let $v(A) = 0$. Then $v(\neg A) = 1$ and hence $v(B) \neq 0$. Therefore, $\models_{\mathbf{LP}} B$. □

Theorem 82 (Soundness)**.** *Let $\mathbf{L} \in \{\mathbf{LP}^{(\circledcirc)_m}_\neg, \mathbf{DG}^{(\circledcirc)_m}_{\mathbf{3}\neg}, \mathbf{K}^{(\circledcirc)_m}_{\mathbf{3}\neg}, \mathbf{G}^{(\circledcirc)_m}_{\mathbf{3}\neg}\}$. Let $\Gamma \subseteq \mathscr{F}^\neg_{(\circledcirc)_m}$ and $A \in \mathscr{F}^\neg_{(\circledcirc)_m}$. Then $\Gamma \vdash_{\mathbf{L}} A$ implies $\Gamma \models_{\mathbf{L}} A$.*

Proof. By induction on the length of the derivation. Use Theorems 76 and 80 as well as Lemma 81. □

Our completeness proof is a generalization of the proofs from [96, 189, 145].

Definition 83. Let $\Gamma \subseteq \mathscr{F}^\neg_{(\circledcirc)_m}$ and $A, B \in \mathscr{F}^\neg_{(\circledcirc)_m}$. Then Γ is said to be an $\mathbf{LP}^{(\circledcirc)_m}_\neg$-*theory* iff the following conditions are fulfilled:

- (Γ_{N}) $\Gamma \neq \mathscr{F}_{(\circledcirc)_m}$ (non-triviality);
- (Γ_{Cl}) $\Gamma \vdash A$ implies $A \in \Gamma$ (closure under $\vdash$);

- (Γ_{M}) $A \in \Gamma$ or $\neg A \in \Gamma$ (maximality);
- $(\Gamma^{0}_{\mathrm{LP}})$ if $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 0$, then $\circledcirc(A_1, \ldots, A_n) \in \Gamma$ and for each $j \in \mathfrak{h}$, $A_j, \neg A_j \in \Gamma$, then for some $i \in \mathfrak{t}$, $\neg A_i \in \Gamma$ or for some $k \in \mathfrak{f}$, $A_k \in \Gamma$;
- $(\Gamma^{1/2}_{\mathrm{LP}})$ if $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 1/2$, then
  - for each $j \in \mathfrak{h}$, $A_j, \neg A_j \in \Gamma$ implies $\circledcirc(A_1, \ldots, A_n) \in \Gamma$, or for some $i \in \mathfrak{t}$, $\neg A_i \in \Gamma$, or for some $k \in \mathfrak{f}$, $A_k \in \Gamma$,
  - for each $j \in \mathfrak{h}$, $A_j, \neg A_j \in \Gamma$ implies $\neg\circledcirc(A_1, \ldots, A_n) \in \Gamma$, or for some $i \in \mathfrak{t}$, $\neg A_i \in \Gamma$, or for some $k \in \mathfrak{f}$, $A_k \in \Gamma$;
- $(\Gamma^{1}_{\mathrm{LP}})$ if $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 1$, then $\neg\circledcirc(A_1, \ldots, A_n) \in \Gamma$ and for each $j \in \mathfrak{h}$, $A_j, \neg A_j \in \Gamma$, then for some $i \in \mathfrak{t}$, $\neg A_i \in \Gamma$ or for some $k \in \mathfrak{f}$, $A_k \in \Gamma$.

Definition 84. An $\mathbf{LP}^{(\circledcirc)_m}_{\neg}$-theory Γ is said to be an $\mathbf{DG}^{(\circledcirc)_m}_{\mathbf{3}_\neg}$-*theory* iff it satisfies the following condition, for each $A \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$:

- (Γ_{RC}) $\neg A \notin \Gamma$ or $\neg\neg A \notin \Gamma$ (restricted consistency).

Definition 85. Let $\Gamma \subseteq \mathscr{F}^{\neg}_{(\circledcirc)_m}$ and $A, B \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$. Then Γ is said to be an $\mathbf{K}^{(\circledcirc)_m}_{\mathbf{3}_\neg}$-*theory* iff the conditions (Γ_{N}) and (Γ_{Cl}) are met as well as the following ones:

- $(\Gamma^{0}_{\mathrm{K}_3})$ if $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 0$, then for each $i \in \mathfrak{t}$, $A_i \in \Gamma$, and for each $k \in \mathfrak{f}$, $\neg A_k \in \Gamma$ implies $\neg\circledcirc(A_1, \ldots, A_n) \in \Gamma$ or for some $j \in \mathfrak{h}$, $A_j \in \Gamma$ or $\neg A_j \in \Gamma$;
- $(\Gamma^{1/2}_{\mathrm{K}_3})$ if $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 1/2$, then
  - $\circledcirc(A_1, \ldots, A_n) \in \Gamma$, for each $i \in \mathfrak{t}$, $A_i \in \Gamma$, and for each $k \in \mathfrak{f}$, $\neg A_k \in \Gamma$ implies for some $j \in \mathfrak{h}$, $A_j \in \Gamma$ or $\neg A_j \in \Gamma$,
  - $\neg\circledcirc(A_1, \ldots, A_n) \in \Gamma$, for each $i \in \mathfrak{t}$, $A_i \in \Gamma$, and for each $k \in \mathfrak{f}$, $\neg A_k \in \Gamma$ implies for some $j \in \mathfrak{h}$, $A_j \in \Gamma$ or $\neg A_j \in \Gamma$;
- $(\Gamma^{1}_{\mathrm{K}_3})$ if $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle) = 1$, then for each $i \in \mathfrak{t}$, $A_i \in \Gamma$, and for each $k \in \mathfrak{f}$, $\neg A_k \in \Gamma$ implies $\circledcirc(A_1, \ldots, A_n) \in \Gamma$ or for some $j \in \mathfrak{h}$, $A_j \in \Gamma$ or $\neg A_j \in \Gamma$.

Lemma 86. *Every* $\mathbf{K}^{(\circledcirc)_m}_{\mathbf{3}_\neg}$*-theory* Γ *satisfies the following condition, for each* $A \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$:

- (Γ_{C}) $A \notin \Gamma$ or $\neg A \notin \Gamma$ (consistency).

Proof. Suppose that (Γ_{C}) does not hold, i.e., there is $A \in \mathscr{F}_{(\circledcirc)_m}$ such that $A \in \Gamma$ and $\neg A \in \Gamma$. Then by the rule (EFQ), $B \in \Gamma$, i.e., $\Gamma = \mathscr{F}_{(\circledcirc)_m}$. However, according to (Γ_{N}), $\Gamma \neq \mathscr{F}_{(\circledcirc)_m}$. Contradiction. Hence, (Γ_{C}) holds. □

Definition 87. An $\mathbf{K}^{(\circledcirc)_m}_{\mathbf{3}_\neg}$-theory Γ is said to be an $\mathbf{G}^{(\circledcirc)_m}_{\mathbf{3}_\neg}$-*theory* iff it satisfies the following condition, for each $A \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$:

- (Γ_{RM}) $\neg A \in \Gamma$ or $\neg\neg A \in \Gamma$ (restricted maximality).

Definition 88. Let $A \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$ and Γ be an $\mathbf{LP}^{(\circledcirc)_m}_{\neg}$- or an $\mathbf{DG}^{(\circledcirc)_m}_{\mathbf{3}_\neg}$-theory. Then $e(A, \Gamma)$ is an *interpretation function* of A in Γ which is defined as follows:

$$e(A, \Gamma) = \begin{cases} 1 & \text{iff} \quad A \in \Gamma, \neg A \notin \Gamma; \\ 1/2 & \text{iff} \quad A, \neg A \in \Gamma; \\ 0 & \text{iff} \quad A \notin \Gamma, \neg A \in \Gamma. \end{cases}$$

Let Γ be an $\mathbf{K}_{\mathbf{3}_\neg}^{(\circledcirc)_m}$- or an $\mathbf{G}_{\mathbf{3}_\neg}^{(\circledcirc)_m}$-theory. Then $e(A,\Gamma)$ is defined as follows:

$$e(A,\Gamma) = \begin{cases} 1 & \text{iff} \quad A \in \Gamma, \neg A \notin \Gamma; \\ 1/2 & \text{iff} \quad A, \neg A \notin \Gamma; \\ 0 & \text{iff} \quad A \notin \Gamma, \neg A \in \Gamma. \end{cases}$$

Lemma 89. *Let* $\mathbf{L} \in \{\mathbf{LP}_\neg^{(\circledcirc)_m}, \mathbf{DG}_{\mathbf{3}_\neg}^{(\circledcirc)_m}, \mathbf{K}_{\mathbf{3}_\neg}^{(\circledcirc)_m}, \mathbf{G}_{\mathbf{3}_\neg}^{(\circledcirc)_m}\}$. *Let* Γ *be an* $\mathbf{L}$*-theory and* $A \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$. *Then*:

(1) $f_\neg(e(A,\Gamma)) = e(\neg A,\Gamma)$;

(2) $f_\circledcirc(e(A_1,\Gamma),\ldots,e(A_n,\Gamma)) = e(\circledcirc(A_1,\ldots,A_n),\Gamma)$.

Proof. The statement (1) is proven in [145]. Let us consider the statement (2).

$\mathbf{L} \in \{\mathbf{LP}_\neg^{(\circledcirc)_m}, \mathbf{DG}_{\mathbf{3}_\neg}^{(\circledcirc)_m}\}$. Let $\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle$ be such that $e(A_i,\Gamma) = 1$, for each $i \in \mathfrak{t}$, $e(A_j,\Gamma) = 1/2$, for each $j \in \mathfrak{h}$, and $e(A_k,\Gamma) = 0$, for each $k \in \mathfrak{f}$. Hence, $A_i \in \Gamma$, $\neg A_i \notin \Gamma$, $A_j \in \Gamma$, $\neg A_j \in \Gamma$, $A_k \notin \Gamma$, $\neg A_k \in \Gamma$, for each $i \in \mathfrak{t}$, $j \in \mathfrak{h}$, and $k \in \mathfrak{f}$.

Suppose that $f_\circledcirc(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle) = 0$. Suppose that $\circledcirc(A_1,\ldots,A_n) \in \Gamma$. By (Γ^0_{LP}), for each $i \in \mathfrak{t}$, $j \in \mathfrak{h}$, and $k \in \mathfrak{f}$, it holds that $\circledcirc(A_1,\ldots,A_n), A_j, \neg A_j \in \Gamma$ implies $\neg A_i \in \Gamma$ or $A_k \in \Gamma$. Since $\circledcirc(A_1,\ldots,A_n), A_j, \neg A_j \in \Gamma$, we have $\neg A_i \in \Gamma$ or $A_k \in \Gamma$. Contradiction. Thus, $\circledcirc(A_1,\ldots,A_n) \notin \Gamma$. By (Γ_{M}), $\neg\circledcirc(A_1,\ldots,A_n) \in \Gamma$. Therefore, for each $i \in \mathfrak{t}$, $j \in \mathfrak{h}$, and $k \in \mathfrak{f}$, $f_\circledcirc(e(A_1,\Gamma),\ldots,e(A_n,\Gamma)) = f_\circledcirc(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle) = f_\circledcirc(\langle e(A_i,\Gamma), e(A_j,\Gamma), e(A_k,\Gamma)\rangle) = 0 = f_\circledcirc(\langle 1, 1/2, 0\rangle) = e(\circledcirc(A_1,\ldots,A_n),\Gamma)$.

Suppose that $f_\circledcirc(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle) = 1/2$. By $(\Gamma^{1/2}_{\mathrm{LP}})$, for each $i \in \mathfrak{t}$, $j \in \mathfrak{h}$, and $k \in \mathfrak{f}$, it holds that $A_j, \neg A_j \in \Gamma$ implies $\circledcirc(A_1,\ldots,A_n) \in \Gamma$ or $\neg A_i \in \Gamma$ or $A_k \in \Gamma$; as well as $A_j, \neg A_j \in \Gamma$ implies $\neg\circledcirc(A_1,\ldots,A_n) \in \Gamma$ or $\neg A_i \in \Gamma$ or $A_k \in \Gamma$. Since $\neg A_i \notin \Gamma$ and $A_k \notin \Gamma$, $\circledcirc(A_1,\ldots,A_n) \in \Gamma$ and $\neg\circledcirc(A_1,\ldots,A_n) \in \Gamma$. Therefore, $f_\circledcirc(e(A_1,\Gamma),\ldots,e(A_n,\Gamma)) = 1/2 = e(\circledcirc(A_1,\ldots,A_n),\Gamma)$.

Suppose that $f_\circledcirc(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle) = 1$. Suppose that $\neg\circledcirc(A_1,\ldots,A_n) \in \Gamma$. By (Γ^1_{LP}), for each $i \in \mathfrak{t}$, $j \in \mathfrak{h}$, and $k \in \mathfrak{f}$, it holds that $\neg\circledcirc(A_1,\ldots,A_n), A_j, \neg A_j \in \Gamma$ implies $\neg A_i \in \Gamma$ or $A_k \in \Gamma$. Contradiction. Hence, $\neg\circledcirc(A_1,\ldots,A_n) \notin \Gamma$. By (Γ_{M}), $\circledcirc(A_1,\ldots,A_n) \in \Gamma$. Therefore, $f_\circledcirc(e(A_1,\Gamma),\ldots,e(A_n,\Gamma)) = 1 = e(\circledcirc(A_1,\ldots,A_n),\Gamma)$.

$\mathbf{L} \in \{\mathbf{K}_{\mathbf{3}_\neg}^{(\circledcirc)_m}, \mathbf{G}_{\mathbf{3}_\neg}^{(\circledcirc)_m}\}$. Let $\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle$ be such that $e(A_i,\Gamma) = 1$, for each $i \in \mathfrak{t}$, $e(A_j,\Gamma) = 1/2$, for each $j \in \mathfrak{h}$, and $e(A_k,\Gamma) = 0$, for each $k \in \mathfrak{f}$. Hence, $A_i \in \Gamma$, $\neg A_i \notin \Gamma$, $A_j \notin \Gamma$, $\neg A_j \notin \Gamma$, $A_k \notin \Gamma$, $\neg A_k \in \Gamma$, for each $i \in \mathfrak{t}$, $j \in \mathfrak{h}$, and $k \in \mathfrak{f}$.

Suppose that $f_\circledcirc(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle) = 0$. By $(\Gamma^0_{\mathrm{K}_3})$, for each $i \in \mathfrak{t}$, $j \in \mathfrak{h}$, and $k \in \mathfrak{f}$, it holds that $A_i, \neg A_k \in \Gamma$ implies $\neg\circledcirc(A_1,\ldots,A_n) \in \Gamma$ or $A_j \in \Gamma$ or $\neg A_j \in \Gamma$. Since $A_i, \neg A_k \in \Gamma$, $\neg\circledcirc(A_1,\ldots,A_n) \in \Gamma$ or $A_j \in \Gamma$ or $\neg A_j \in \Gamma$. Since $A_j \notin \Gamma$ and $\neg A_j \notin \Gamma$, $\neg\circledcirc(A_1,\ldots,A_n) \in \Gamma$. By (Γ_{C}), $\circledcirc(A_1,\ldots,A_n) \notin \Gamma$. Therefore, $f_\circledcirc(e(A_1,\Gamma),\ldots,e(A_n,\Gamma)) = 0 = e(\circledcirc(A_1,\ldots,A_n),\Gamma)$.

Suppose that $f_\circledcirc(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle) = 1/2$. By $(\Gamma^{1/2}_{\mathrm{K}_3})$, for each $i \in \mathfrak{t}$, $j \in \mathfrak{h}$, and $k \in \mathfrak{f}$, it holds that $\circledcirc(A_1,\ldots,A_n), A_i, \neg A_k \in \Gamma$ implies $A_j \in \Gamma$ or $\neg A_j \in \Gamma$, as well as $\neg\circledcirc(A_1,\ldots,A_n), A_i, \neg A_k \in \Gamma$ implies $A_j \in \Gamma$ or $\neg A_j \in \Gamma$. Since $A_j \notin \Gamma$ and $\neg A_j \notin \Gamma$, $(\circledcirc(A_1,\ldots,A_n) \notin \Gamma$ and $\neg\circledcirc(A_1,\ldots,A_n) \notin \Gamma)$ or $A_i \notin \Gamma$ or $\neg A_k \notin \Gamma$ Since $A_i, \neg A_k \in \Gamma$, $\circledcirc(A_1,\ldots,A_n) \notin \Gamma$ and $\neg\circledcirc(A_1,\ldots,A_n) \notin \Gamma$. Therefore, $f_\circledcirc(e(A_1,\Gamma),\ldots,e(A_n,\Gamma)) = 1/2 = e(\circledcirc(A_1,\ldots,A_n),\Gamma)$.

Suppose that $f_\circledcirc(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle) = 1$. By $(\Gamma^1_{\mathrm{K}_3})$, for each $i \in \mathfrak{t}$, $j \in \mathfrak{h}$, and $k \in \mathfrak{f}$, it holds that $A_i, \neg A_k \in \Gamma$ implies $\circledcirc(A_1,\ldots,A_n) \in \Gamma$ or $A_j \in \Gamma$ or $\neg A_j \in \Gamma$. Hence, $\circledcirc(A_1,\ldots,A_n) \in \Gamma$. By (Γ_{C}), $\neg\circledcirc(A_1,\ldots,A_n) \notin \Gamma$. Therefore, $f_\circledcirc(e(A_1,\Gamma),\ldots,e(A_n,\Gamma)) = 1 = e(\circledcirc(A_1,\ldots,A_n),\Gamma)$. □

Definition 90. Let $\mathbf{L} \in \{\mathbf{LP}_\neg^{(\circledcirc)_m}, \mathbf{DG}_{\mathbf{3}_\neg}^{(\circledcirc)_m}, \mathbf{K}_{\mathbf{3}_\neg}^{(\circledcirc)_m}, \mathbf{G}_{\mathbf{3}_\neg}^{(\circledcirc)_m}\}$. Let Γ be an $\mathbf{L}$-theory and $p \in \mathcal{P}$. Then v_Γ is said to be a *canonical valuation* iff $v_\Gamma(p) = e(p,\Gamma)$.

Lemma 91. *Let* $\mathbf{L} \in \{\mathbf{LP}_\neg^{(\circledcirc)_m}, \mathbf{DG}_{\mathbf{3}_\neg}^{(\circledcirc)_m}, \mathbf{K}_{\mathbf{3}_\neg}^{(\circledcirc)_m}, \mathbf{G}_{\mathbf{3}_\neg}^{(\circledcirc)_m}\}$. *Let* Γ *be an* $\mathbf{L}$*-theory,* $A \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$, *and* v_Γ *be a canonical valuation. Then* $v_\Gamma(A) = e(A,\Gamma)$.

Proof. By induction on A. Use Lemma 89. □

Lemma 92 (Lindenbaum). *Let* $\mathbf{L} \in \{\mathbf{LP}_{\neg}^{(\circledcirc)m}, \mathbf{DG}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}, \mathbf{K}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}, \mathbf{G}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}\}$. *Let* $\Gamma \subseteq \mathscr{F}^{\neg}_{(\circledcirc)m}$, $A \in \mathscr{F}^{\neg}_{(\circledcirc)m}$, *and* $\Gamma \nvdash_{\mathbf{L}} A$. *Then there exists an* $\mathbf{L}$*-theory* Δ *such that* $\Gamma \subseteq \Delta$ *and* $\Delta \nvdash_{\mathbf{L}} A$.

Proof. By standard method of the proof of Lindenbaum's Lemma (see, for example, [96, 145]). Assume that $\Gamma \nvdash_{\mathbf{L}} A$. Let $D_1, \ldots, D_h, \ldots$ be an enumeration of all the members of $\mathscr{F}^{\neg}_{(\circledcirc)m}$. Let us define Δ as follows, where $\Theta_0 \subseteq \mathscr{F}^{\neg}_{(\circledcirc)m}, \ldots, \Theta_l \subseteq \mathscr{F}^{\neg}_{(\circledcirc)m}, \ldots$:

$$\Theta_0 = \Gamma$$

$$\Theta_{t+1} = \begin{cases} \Theta_t \cup \{D_{t+1}\}, & \text{if } \Theta_t \cup \{D_{t+1}\} \nvdash A; \\ \Theta_t & \text{otherwise;} \end{cases}$$

$$\Delta = \bigcup_{t=0}^{\infty} \Theta_t.$$

By the definition of Δ, we have $\Gamma \subseteq \Delta$. By a straightforward induction on i, one can easily prove that $\Delta \nvdash_{\mathbf{L}} A$. Since $\Delta \nvdash_{\mathbf{L}} A$, Δ satisfies (Γ_{N}). By the transitivity of the consequence relation, we obtain that Δ satisfies (Γ_{Cl}).

Suppose that $\mathbf{L} = \mathbf{LP}_{\neg}^{(\circledcirc)m}$. Let us prove that Δ satisfies (Γ_{M}). Suppose that for some $B \in \mathscr{F}^{\neg}_{(\circledcirc)m}$, it holds that $B \notin \Delta$ and $\neg B \notin \Delta$. Then there are D_e and D_g such that $D_e = B$ and $D_g = \neg B$, $\Theta_{e-1} \cup \{D_e\} \vdash_{\mathbf{L}} A$ and $\Theta_{g-1} \cup \{D_g\} \vdash_{\mathbf{L}} A$. Then $\Delta \cup \{D_e\} \vdash_{\mathbf{L}} A$ and $\Delta \cup \{D_g\} \vdash_{\mathbf{L}} A$, since $\Theta_{e-1} \subseteq \Delta$ and $\Theta_{g-1} \subseteq \Delta$. By the rule (EM), $\Delta \vdash_{\mathbf{L}} A$. Contradiction. Thus, for any $B \in \mathscr{F}^{\neg}_{(\circledcirc)m}$, it holds that $B \in \Delta$ or $\neg B \in \Delta$.

Let us prove that Δ satisfies (Γ^0_{LP}). Assume that for some $B_1, \ldots, B_n \in \mathscr{F}^{\neg}_{(\circledcirc)m}$, $\circledcirc(B_1, \ldots, B_n) \in \Delta$, for each $j \in \mathfrak{h}$, $B_j, \neg B_j \in \Delta$, while for each $i \in \mathfrak{t}$, $\neg B_i \notin \Delta$ and for each $k \in \mathfrak{f}$, $B_k \notin \Delta$. Then there are $D_e = \neg B_i$ and $D_g = B_k$ such that $\Theta_{e-1} \cup \{D_e\} \vdash_{\mathbf{L}} A$ and $\Theta_{g-1} \cup \{D_g\} \vdash_{\mathbf{L}} A$. Then $\Delta \cup \{D_e\} \vdash_{\mathbf{L}} A$ and $\Delta \cup \{D_g\} \vdash_{\mathbf{L}} A$. By the rule $R_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle 0)$, $\Delta \vdash_{\mathbf{L}} A$. Contradiction. Thus, $\circledcirc(A_1, \ldots, A_n) \in \Gamma$, for each $j \in \mathfrak{h}$, $A_j, \neg A_j \in \Gamma$ implies for some $i \in \mathfrak{t}$, $\neg A_i \in \Gamma$ or for some $k \in \mathfrak{f}$, $A_k \in \Gamma$.

The other cases are considered similarly. $\square$

Theorem 93 (Completeness). *Let* $\mathbf{L} \in \{\mathbf{LP}_{\neg}^{(\circledcirc)m}, \mathbf{DG}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}, \mathbf{K}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}, \mathbf{G}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}\}$. *Let* $\Gamma \subseteq \mathscr{F}^{\neg}_{(\circledcirc)m}$ *and* $A \in \mathscr{F}^{\neg}_{(\circledcirc)m}$. *Then* $\Gamma \models_{\mathbf{L}} A$ *implies* $\Gamma \vdash_{\mathbf{L}} A$.

Proof. By contraposition. Let $\mathbf{L} \in \{\mathbf{LP}_{\neg}^{(\circledcirc)m}, \mathbf{DG}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}, \mathbf{K}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}, \mathbf{G}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}\}$. Suppose $\Gamma \nvdash_{\mathbf{L}} A$. By Lemma 92, there is an $\mathbf{L}$-theory Δ such that $\Gamma \subseteq \Delta$ and $\Delta \nvdash_{\mathbf{L}} A$. Let v_Δ be a canonical valuation introduced in Definition 90. Let $\mathbf{L} \in \{\mathbf{LP}_{\neg}^{(\circledcirc)m}, \mathbf{DG}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}\}$. By Lemma 91, $v_\Delta(B) \neq 0$, for each $B \in \Gamma$, while $v(A) = 0$. Thus, $\Gamma \not\models_{\mathbf{L}} A$. Let $\mathbf{L} \in \{\mathbf{K}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}, \mathbf{G}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}\}$. By Lemma 91, $v_\Delta(B) = 1$, for each $B \in \Gamma$, while $v(A) \neq 1$. Thus, $\Gamma \not\models_{\mathbf{L}} A$. $\square$

Corollary 94 (Adequacy). *Let* $\mathbf{L} \in \{\mathbf{LP}_{\neg}^{(\circledcirc)m}, \mathbf{DG}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}, \mathbf{K}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}, \mathbf{G}_{\mathbf{3}_{\neg}}^{(\circledcirc)m}\}$. *Let* $\Gamma \subseteq \mathscr{F}^{\neg}_{(\circledcirc)m}$ *and* $A \in \mathscr{F}^{\neg}_{(\circledcirc)m}$. *Then* $\Gamma \models_{\mathbf{L}} A$ *iff* $\Gamma \vdash_{\mathbf{L}} A$.

Proof. Follows from Theorems 82 and 93. $\square$

3.4 Proof of the normalisation theorem

Following [104, Definition 4], we understand the notion of a maximal formula in the subsequent way.

Definition 95 (Maximal formula). A *maximal formula* with the main operator $\neg$ or $\circledcirc$ in a deduction in one of the logics in question is an occurrence of a formula $\neg A$ or $\circledcirc(A_1, \ldots, A_n)$ that is the major premise of an application of a general elimination rule for $\neg$ or $\circledcirc$ and the major assumption discharged by an application of a general introduction rule for $\neg$ or $\circledcirc$.

Note that among maximal formulas of the form $\neg A$ there may be formulas $\neg\circledcirc(A_1, \dots, A_n)$ and $\neg\neg B$.

Definition 96 (Degree of a formula)**.** We define a degree d of a formula A inductively as follows, where p is an atomic formula:

- $d(p) = d(\neg p) = 1$,
- if $A \neq p$, then $d(\neg A) = d(A) + 1$,
- $d(\circledcirc(A_1, \dots, A_n)) = \sum_{i=1}^{i=n} d(A_i) + 2$.

One of the features of this definition is that if $\circledcirc$ is a unary connective, then $\circledcirc A$ has a higher degree than $\neg A$ which is important for the forthcoming reduction and permutation procedures. In the case of unary connectives, as some of the rules of an inference discharge negated subformulas of major premises or assumptions, it can happen that the procedure introduces a maximal formula that is a negated subformula of the maximal formula to be removed. If the main connective in question is unary, then determining the complexity of formulas as usual by counting connectives one each would introduce a maximal formula of the same degree as the removed maximal formula. Our measure of complexity avoids this. Another property is that literals have the same degree which is also important, since we are using negated formulas in the simplified reductions which justify the negation subformula property.

The notions of a segment, its length and degree, a maximal segment, a deduction in normal form, and a rank of a deduction are understood according to Definitions 51, 52, 53, 54, and 55,

We begin our proof with an example of simplicity conversions which can remove applications of rules with vacuous discharge above arbitrary premises. Since these conversions are rather obvious, we give just one example.

$$R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 0\big)\ \dfrac{\begin{matrix}\mathfrak{D}_1 \\ \circledcirc(\vec{A})\end{matrix} \quad \begin{matrix}\mathfrak{D}_2^{\dagger} \\ B\end{matrix} \quad \begin{matrix}\mathfrak{D}_3^{\ddagger} \\ A_j^{\ddagger}\end{matrix} \quad \begin{matrix}\mathfrak{D}_4^{\ddagger} \\ \neg A_j^{\ddagger}\end{matrix} \quad \begin{matrix}\mathfrak{D}_5^{*} \\ B\end{matrix}}{B} \quad \dashrightarrow \quad \begin{matrix}\mathfrak{D}_2^{\dagger} \\ B\end{matrix}$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$.

Also, before we start the main part of the proof, let us consider the case of maximal segments described in the second clause of Definition 51. We illustrate it under example of two applications of the rule $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 0\big)$ such that $\mathfrak{t} = \mathfrak{f} = \emptyset$ (the other cases are similar, although in the proof for $\mathbf{K_3}$ we will pay a special attention to the case with (EFQ)).

$$R_{\circledcirc}\big(\langle \emptyset, \mathfrak{h}', \emptyset\rangle 0\big)\ \dfrac{R_{\circledcirc}\big(\langle \emptyset, \mathfrak{h}, \emptyset\rangle 0\big)\ \dfrac{\begin{matrix}\mathfrak{D}_1 \\ \circledcirc(\vec{A})\end{matrix} \quad \begin{matrix}\mathfrak{D}_2^{\dagger} \\ A_j^{\dagger}\end{matrix} \quad \begin{matrix}\mathfrak{D}_3^{\dagger} \\ \neg A_j^{\dagger}\end{matrix}}{\circledcirc(\vec{B})} \quad \begin{matrix}\mathfrak{D}_4^{\ddagger} \\ B_j^{\ddagger}\end{matrix} \quad \begin{matrix}\mathfrak{D}_5^{\ddagger} \\ \neg B_j^{\ddagger}\end{matrix}}{C} \quad \dashrightarrow \quad R_{\circledcirc}\big(\langle \emptyset, \mathfrak{h}, \emptyset\rangle 0\big)\ \dfrac{\begin{matrix}\mathfrak{D}_1 \\ \circledcirc(\vec{A})\end{matrix} \quad \begin{matrix}\mathfrak{D}_2^{\dagger} \\ A_j^{\dagger}\end{matrix} \quad \begin{matrix}\mathfrak{D}_3^{\dagger} \\ \neg A_j^{\dagger}\end{matrix}}{C}$$

† for each $j \in \mathfrak{h}$, ‡ for each $j \in \mathfrak{h}'$.

3.4.1 The case of LP: reduction procedures

The next lemma, which helps in the normalisation proof, establishes that (EM) can be restricted to discharging only one major assumption, i.e., an assumption of the form $\neg\alpha$ above its right premise, that is, the assumption class of major assumptions discharged is a singleton. (The case where it is empty may be excluded: clearly such an application is superfluous and may be removed from deductions following the usual pattern of simplification conversions).

In essence, the proof proceeds by multiplying an application of (EM) that discharges more than one major assumption so that a sequence of application of (EM) results where each of which discharges only one. It needs to be ensured, however, that applications of (EM) that discharge major assumptions that occur above the left premise of the application of (EM) to be treated are not multiplied. The procedure needs to ensure that it does not introduce more and more applications of (EM) that discharge more than one major assumption. To handle this case, we need a definition to capture the following possibility. Above the left premise of an application of (EM), other applications of (EM) discharge major assumptions, above the left premises of which yet other applications of (EM) also discharge major assumptions, and so on. In any such sequence of applications of (EM) there will be the first and the last. There may also be further applications of rules of inference interspersed, but these need not concern us.

Definition 97 ((EM)-chain)**.** A *chain* of applications of (EM), $(\mathrm{EM})_1$, $(\mathrm{EM})_2$ ... $(\mathrm{EM})_n$, (an (EM)-*chain* for short) is a sequence of applications of (EM) such that above the right premise of $(\mathrm{EM})_1$ no assumptions are discharged that stand above the left assumption of another application of (EM) (i.e., no minor assumptions discharged by another application of (EM)), above the left premise of $(\mathrm{EM})_1$ there are such assumptions (i.e., major assumptions discharged by other applications of (EM)); for every $i < n$, there is a j such that $i + j \leq n$, $(\mathrm{EM})_{i+j}$ discharges assumptions above its right premise (i.e., its major assumptions) that stand above the left premise of $(\mathrm{EM})_i$; and above the left premise of $(\mathrm{EM})_n$ no assumptions are discharged that stand above the right premise of another application of (EM) (i.e., no major assumptions of other applications of (EM)).

Lemma 98. *Any deduction in* $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$ (*resp.* $\mathbf{DG}_{\mathbf{3}_\neg}^{(\circledcirc)_m}$) *can be transformed into one in which every application of the rule* (EM) *discharges exactly one major assumption.*

Proof. First, by the ban of vacuous discharge, every application of (EM) discharges at least one major assumption. Consider the *last* application of (EM) of an (EM)-chain that discharges more than one major assumption (i.e., the assumption class of which contains more than one formula), say it discharges l formula occurrences of the type $\neg A$:

$$(\mathrm{EM})^{a,b}\ \dfrac{\begin{matrix}[A]^a\\ \mathfrak{D}\\ B\end{matrix} \qquad \begin{matrix}\underbrace{[\neg A]^b, [\neg A]^b, \dots, [\neg A]^b}\\ \mathfrak{E}\\ B\end{matrix}}{B}$$

Instead of making this one application of (EM), one can apply it l times:

$$(\mathrm{EM})^{a,l}\ \dfrac{\begin{matrix}[A]^a\\ \mathfrak{D}\\ B\end{matrix} \qquad \begin{matrix} (\mathrm{EM})^{a,2}\ \dfrac{\begin{matrix}[A]^a\\ \mathfrak{D}\\ B\end{matrix} \qquad (\mathrm{EM})^{a,1}\ \dfrac{\begin{matrix}[A]^a\\ \mathfrak{D}\\ B\end{matrix} \qquad \begin{matrix}\underbrace{[\neg A]^1, [\neg A]^2, \dots, [\neg A]^l}\\ \mathfrak{E}\\ B\end{matrix}}{B}}{B} \\ \vdots \\ B\end{matrix}}{B}$$

As $(\mathrm{EM})^{a,b}$ is the last application of the (EM)-chain, no further major assumptions of (EM) are discharged in $\mathfrak{D}$ in the original deduction, and hence none are multiplied in the reduced deduction.

It remains to find a suitable induction measure: the number of applications of (EM) that discharge more than one major assumption in a deduction suffices: this number has been reduced by one by the above procedure. For systematicity in applying the procedure, apply the procedure to an (EM)-chain such that no other (EM)-chain stands below it in the deduction. □

The negation fragment of LP

Maximal formulas. Case 1. The maximal formula has the form $\neg\neg A$; the rules $(\neg\neg I)$ and $(\neg\neg E)$ are applied. Convert the deduction on the left into the deduction on the right (the symbol $\dashrightarrow$ stands for the replacement).

$$(\neg\neg I)^a\,\dfrac{\begin{array}{c}\mathfrak{D}\\ A\end{array}\qquad \begin{array}{c}(\neg\neg E)^b\,\dfrac{[\neg\neg A]^a\quad \begin{array}{c}[A]^b\\ \mathfrak{E}_1\\ B\end{array}}{B}\\ \mathfrak{E}_2\\ C\end{array}}{C}\quad\dashrightarrow\quad (\neg\neg I)^a\,\dfrac{\begin{array}{c}\mathfrak{D}\\ A\end{array}\quad \begin{array}{c}\mathfrak{D}\\ A\\ \mathfrak{E}_1\\ B\\ \mathfrak{E}_2\\ C\end{array}}{C}$$

Case 2. The maximal formula has the form $\neg\neg A$; the rules (EM) and $(\neg\neg E)$ are applied.

$$(\mathrm{EM})^{a,b}\,\dfrac{\begin{array}{c}[\neg A]^a\\ \mathfrak{D}_1\\ B\end{array}\qquad \begin{array}{c}(\neg\neg E)^c\,\dfrac{[\neg\neg A]^b\quad \begin{array}{c}[A]^c\\ \mathfrak{D}_2\\ C\end{array}}{C}\\ \mathfrak{D}_3\\ B\end{array}}{B}\quad\dashrightarrow\quad (\mathrm{EM})^{c,a}\,\dfrac{\begin{array}{c}[A]^c\\ \mathfrak{D}_2\\ C\\ \mathfrak{D}_3\\ B\end{array}\quad \begin{array}{c}[\neg A]^a\\ \mathfrak{D}_1\\ B\end{array}}{B}$$

Note that due to Lemma 98 there is only one formula in the assumption class b (Lemma 98 has to be applied before doing this reduction).

Maximal segments. Case 3. The major premise of $(\neg\neg E)$ is derived by $(\neg\neg I)$.

$$(\neg\neg E)^b\,\dfrac{(\neg\neg I)^a\,\dfrac{\begin{array}{c}\mathfrak{D}_1\\ A\end{array}\quad \begin{array}{c}[\neg\neg A]^a\\ \mathfrak{D}_2\\ \neg\neg B\end{array}}{\neg\neg B}\qquad \begin{array}{c}[B]^b\\ \mathfrak{D}_3\\ C\end{array}}{C}\quad\dashrightarrow\quad (\neg\neg I)^a\,\dfrac{\begin{array}{c}\mathfrak{D}_1\\ A\end{array}\quad (\neg\neg E)^b\,\dfrac{\begin{array}{c}[\neg\neg A]^a\\ \mathfrak{D}_2\\ \neg\neg B\end{array}\quad \begin{array}{c}[B]^b\\ \mathfrak{D}_3\\ C\end{array}}{C}}{C}$$

Notice that B may coincide with A. Then we have a simpler situation (in what follows, we will not emphasize such cases).

$$(\neg\neg E)^b\,\dfrac{(\neg\neg I)^a\,\dfrac{\begin{array}{c}\mathfrak{D}\\ A\end{array}\quad [\neg\neg A]^a}{\neg\neg A}\qquad \begin{array}{c}[A]^b\\ \mathfrak{E}\\ C\end{array}}{C}\quad\dashrightarrow\quad \begin{array}{c}\mathfrak{D}\\ A\\ \mathfrak{E}\\ C\end{array}$$

Case 4. The major premise of $(\neg\neg E)$ is derived by (EM).

$$(\neg\neg E)^c\,\dfrac{(\mathrm{EM})^{a,b}\,\dfrac{\begin{array}{c}[B]^a\\ \mathfrak{D}_1\\ \neg\neg A\end{array}\quad \begin{array}{c}[\neg B]^b\\ \mathfrak{D}_2\\ \neg\neg A\end{array}}{\neg\neg A}\qquad \begin{array}{c}[A]^c\\ \mathfrak{D}_3\\ C\end{array}}{C}\quad\dashrightarrow\quad (\mathrm{EM})^{a,b}\,\dfrac{(\neg\neg E)^c\,\dfrac{\begin{array}{c}[B]^a\\ \mathfrak{D}_1\\ \neg\neg A\end{array}\quad \begin{array}{c}[A]^c\\ \mathfrak{D}_3\\ C\end{array}}{C}\qquad (\neg\neg E)^c\,\dfrac{\begin{array}{c}[\neg B]^b\\ \mathfrak{D}_2\\ \neg\neg A\end{array}\quad \begin{array}{c}[A]^c\\ \mathfrak{D}_3\\ C\end{array}}{C}}{C}$$

The negation fragment of LP extended by n-ary operators $\circledcirc_1, \ldots, \circledcirc_m$

Maximal formulas. Case 1. The maximal formula of the form $\neg\circledcirc(\vec{A})$ is produced by applications of the rules $R_\circledcirc(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1)$ and (EM) ($\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively, stand for them).

$$\mathfrak{R}_2^{a,b}\,\dfrac{\begin{array}{c}[\circledcirc(\vec{A})]^a\\ \mathfrak{D}_1\\ C\end{array} \qquad \begin{array}{c} \mathfrak{R}_1^{c,d}\,\dfrac{[\neg\circledcirc(\vec{A})]^b \quad \begin{array}{c}[\neg A_i^\dagger]^c\\ \mathfrak{D}_2^\dagger\\ B\end{array} \quad \begin{array}{c}\mathfrak{D}_3^\ddagger\\ A_j^\ddagger\end{array} \quad \begin{array}{c}\mathfrak{D}_4^\ddagger\\ \neg A_j^\ddagger\end{array} \quad \begin{array}{c}[A_k^*]^d\\ \mathfrak{D}_5^*\\ B\end{array}}{B} \\ \mathfrak{E}\\ C\end{array}}{C}$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$.

To eliminate the maximal formula we need to replace the above derivation with the following one, containing the application of the rule $R_\circledcirc(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2)$ (we abbreviate it as $\mathfrak{R}_3$). Recall that it is derivable in the presence of $R_\circledcirc(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1)$ and (EM) (see Proposition 78), but since this derivation is itself not normal we postulated that $R_\circledcirc(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2)$ has to be a primitive rule in a system in order to prove normalisation.

$$\mathfrak{R}_3^{a,c,d}\,\dfrac{\begin{array}{c}[\circledcirc(\vec{A})]^a\\ \mathfrak{D}_1\\ C\end{array} \quad \begin{array}{c}[\neg A_i^\dagger]^c\\ \mathfrak{D}_2^\dagger\\ B\\ \mathfrak{E}\\ C\end{array} \quad \begin{array}{c}\mathfrak{D}_3^\ddagger\\ A_j^\ddagger\end{array} \quad \begin{array}{c}\mathfrak{D}_4^\ddagger\\ \neg A_j^\ddagger\end{array} \quad \begin{array}{c}[A_k^*]^d\\ \mathfrak{D}_5^*\\ B\\ \mathfrak{E}\\ C\end{array}}{C}$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$.

Note that due to Lemma 98 there is only one formula in the assumption class b (Lemma 98 has to be applied before doing this reduction).

Case 2. *Subcase 2.1.* The maximal formula $\circledcirc(\vec{A})$ produced by applications of the rules $R_\circledcirc(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0)$ and $R_\circledcirc(\langle\mathfrak{t}',\mathfrak{h}',\mathfrak{f}'\rangle 1/2)$ which we denote by $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively.

$$\mathfrak{R}_2^{a,d,e}\,\dfrac{\begin{array}{c}\mathfrak{R}_1^{b,c}\,\dfrac{[\circledcirc(\vec{A})]^a \quad \begin{array}{c}[\neg A_i^\dagger]^b\\ \mathfrak{D}_1^\dagger\\ B\end{array} \quad \begin{array}{c}\mathfrak{D}_2^\ddagger\\ A_j^\ddagger\end{array} \quad \begin{array}{c}\mathfrak{D}_3^\ddagger\\ \neg A_j^\ddagger\end{array} \quad \begin{array}{c}[A_k^*]^c\\ \mathfrak{D}_4^*\\ B\end{array}}{B}\\ \mathfrak{E}\\ C\end{array} \quad \begin{array}{c}[\neg A_{i'}^\sharp]^d\\ \mathfrak{D}_5^\sharp\\ C\end{array} \quad \begin{array}{c}\mathfrak{D}_6^\S\\ A_{j'}^\S\end{array} \quad \begin{array}{c}\mathfrak{D}_7^\S\\ \neg A_{j'}^\S\end{array} \quad \begin{array}{c}[A_{k'}^\star]^e\\ \mathfrak{D}_8^\star\\ C\end{array}}{C}$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$; $^\sharp$for each $i' \in \mathfrak{t}'$, §for each $j' \in \mathfrak{h}'$, *for each $k' \in \mathfrak{f}'$.

This case is the most complicated in our proof. First of all, let us recall that all the formulas $\neg A_i, A_j, \neg A_j, A_k, \neg A_{i'}, A_{j'}, \neg A_{j'}, A_{k'}$ are either subformulas or negations of subformulas of $\circledcirc(\vec{A}) = \circledcirc(A_1, \ldots, A_n)$. We need to have more details about the other rules which are present in the system. Let us observe that this case can be reformulated in the following way.

$$\mathfrak{R}_2^{a,d,e}\,\dfrac{\begin{array}{c}\mathfrak{R}_1^{b,c}\,\dfrac{[\circledcirc(\vec{A})]^a \qquad X_1 \qquad \dots \qquad X_n}{B} \\ \mathfrak{E} \\ C\end{array} \qquad\qquad Y_1 \qquad \dots \qquad Y_n}{C}$$

where $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are abbreviations for the following derivations, for any $t, u \in \{1, \dots, n\}$.

$$X_t = \begin{cases} \begin{array}{c}[A_t]^b \\ \mathfrak{D}_1 \\ B\end{array} & \textit{iff} \quad t \in \mathfrak{f}; \\[2ex] \begin{array}{cc}\mathfrak{D}_2 & \mathfrak{D}_3 \\ A_t & \neg A_t\end{array} & \textit{iff} \quad t \in \mathfrak{h}; \\[2ex] \begin{array}{c}[\neg A_t]^c \\ \mathfrak{D}_4 \\ B\end{array} & \textit{iff} \quad t \in \mathfrak{t}; \end{cases} \qquad Y_u = \begin{cases} \begin{array}{c}[A_u]^d \\ \mathfrak{D}_5 \\ C\end{array} & \textit{iff} \quad u \in \mathfrak{f}'; \\[2ex] \begin{array}{cc}\mathfrak{D}_6 & \mathfrak{D}_7 \\ A_u & \neg A_u\end{array} & \textit{iff} \quad u \in \mathfrak{h}'; \\[2ex] \begin{array}{c}[\neg A_u]^e \\ \mathfrak{D}_8 \\ C\end{array} & \textit{iff} \quad u \in \mathfrak{t}'. \end{cases}$$

As follows from these equalities and soundness of our natural deduction systems, there is $l \in \{1, \dots, n\}$ such that $X_l \neq Y_l$.[7] Then the following combinations are possible (we present them in the form of ordered pairs $\langle X_l, Y_l \rangle$):

$$\mathcal{C}_1 = \left\langle \begin{array}{c}[A_l]^b \\ \mathfrak{D}_1 \\ B\end{array}, \begin{array}{cc}\mathfrak{D}_6 & \mathfrak{D}_7 \\ A_l & \neg A_l\end{array} \right\rangle \quad \mathcal{C}_2 = \left\langle \begin{array}{c}[A_l]^b \\ \mathfrak{D}_1 \\ B\end{array}, \begin{array}{c}[\neg A_l]^e \\ \mathfrak{D}_8 \\ C\end{array} \right\rangle$$

$$\mathcal{C}_3 = \left\langle \begin{array}{cc}\mathfrak{D}_2 & \mathfrak{D}_3 \\ A_l & \neg A_l\end{array}, \begin{array}{c}[A_l]^d \\ \mathfrak{D}_5 \\ C\end{array} \right\rangle \quad \mathcal{C}_4 = \left\langle \begin{array}{cc}\mathfrak{D}_2 & \mathfrak{D}_3 \\ A_l & \neg A_l\end{array}, \begin{array}{c}[\neg A_l]^e \\ \mathfrak{D}_8 \\ C\end{array} \right\rangle$$

$$\mathcal{C}_5 = \left\langle \begin{array}{c}[\neg A_l]^c \\ \mathfrak{D}_4 \\ B\end{array}, \begin{array}{c}[A_l]^d \\ \mathfrak{D}_5 \\ C\end{array} \right\rangle \quad \mathcal{C}_6 = \left\langle \begin{array}{c}[\neg A_l]^c \\ \mathfrak{D}_4 \\ B\end{array}, \begin{array}{cc}\mathfrak{D}_6 & \mathfrak{D}_7 \\ A_l & \neg A_l\end{array} \right\rangle$$

Let us consider the case $\mathcal{C}_1$.

$$\mathfrak{R}_2^{a,d,e}\,\dfrac{\begin{array}{c}\mathfrak{R}_1^{b,c}\,\dfrac{[\circledcirc(\vec{A})]^a \qquad X_1 \dots X_{l-1} \qquad \begin{array}{c}[A_l]^b \\ \mathfrak{D}_1 \\ B\end{array} \qquad X_{l+1} \dots X_n}{B} \\ \mathfrak{E} \\ C\end{array} \qquad Y_1 \dots Y_{l-1} \qquad \begin{array}{c}\mathfrak{D}_6 \\ A_l\end{array} \qquad \begin{array}{c}\mathfrak{D}_7 \\ \neg A_l\end{array} \qquad Y_{l+1} \dots Y_n}{C}$$

Then we can introduce the following reduction procedure:

$$\mathfrak{R}_2^{a,d,e}\,\dfrac{\begin{array}{c}\mathfrak{D}_6 \\ A_l \\ \mathfrak{D}_1 \\ B \\ \mathfrak{E} \\ C\end{array} \qquad Y_1 \dots Y_{l-1} \qquad \begin{array}{c}\mathfrak{D}_6 \\ A_l\end{array} \qquad \begin{array}{c}\mathfrak{D}_7 \\ \neg A_l\end{array} \qquad Y_{l+1} \dots Y_n}{C}$$

[7] Otherwise $\mathfrak{t} = \mathfrak{t}'$, $\mathfrak{h} = \mathfrak{h}'$, and $\mathfrak{f} = \mathfrak{f}'$ which implies that both $f_\circledcirc\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle\big) = 0$ and $f_\circledcirc\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f} \rangle\big) = 1/2$.

Let us consider the case $\mathcal{C}_2$.

$$\mathfrak{R}_2^{a,d,e}\,\dfrac{\begin{array}{c}\mathfrak{R}_1^{b,c}\,\dfrac{[\circledcirc(\vec{A})]^a \quad X_1\ldots X_{l-1} \quad \begin{array}{c}[A_l]^b\\ \mathfrak{D}_1\\ B\end{array} \quad X_{l+1}\ldots X_n}{B}\\ \mathfrak{E}\\ C\end{array} \quad Y_1\ldots Y_{l-1} \quad \begin{array}{c}[\neg A_l]^e\\ \mathfrak{D}_8\\ C\end{array} \quad Y_{l+1}\ldots Y_n}{C}$$

Then we can introduce the following reduction procedure:

$$\mathfrak{R}_2^{a,d,e}\,\dfrac{\begin{array}{c}(\mathrm{EM})^{b,e}\,\dfrac{\begin{array}{c}[A_l]^b\\ \mathfrak{D}_1\\ B\end{array} \quad \begin{array}{c}[\neg A_l]^e\\ \mathfrak{D}_8\\ B\end{array}}{B}\\ \mathfrak{E}\\ C\end{array} \quad Y_1\ldots Y_{l-1} \quad \begin{array}{c}[\neg A_l]^e\\ \mathfrak{D}_8\\ C\end{array} \quad Y_{l+1}\ldots Y_n}{C}$$

The case $\mathcal{C}_5$ is considered similarly.
Let us consider the case $\mathcal{C}_3$.

$$\mathfrak{R}_2^{a,d,e}\,\dfrac{\begin{array}{c}\mathfrak{R}_1^{b,c}\,\dfrac{[\circledcirc(\vec{A})]^a \quad X_1\ldots X_{l-1} \quad \begin{array}{c}\mathfrak{D}_2\\ A_l\end{array} \quad \begin{array}{c}\mathfrak{D}_3\\ \neg A_l\end{array} \quad X_{l+1}\ldots X_n}{B}\\ \mathfrak{E}\\ C\end{array} \quad Y_1\ldots Y_{l-1} \quad \begin{array}{c}[A_l]^d\\ \mathfrak{D}_5\\ C\end{array} \quad Y_{l+1}\ldots Y_n}{C}$$

Then we can introduce the following reduction procedure:

$$\mathfrak{R}_2^{a,d,e}\,\dfrac{\begin{array}{c}\mathfrak{D}_2\\ A_l\\ \mathfrak{D}_5\\ C\\ \mathfrak{E}^*\\ C\end{array} \quad Y_1\ldots Y_{l-1} \quad \begin{array}{c}[A_l]^d\\ \mathfrak{D}_5\\ C\end{array} \quad Y_{l+1}\ldots Y_n}{C}$$

where $\mathfrak{E}^*$ is defined via the ρ-reduction described in the normalisation proof for Segerberg's system on p. 55.
Let us consider the case $\mathcal{C}_4$.

$$\mathfrak{R}_1^{b,c} \qquad \mathfrak{R}_2^{a,d,e}\,\dfrac{\begin{array}{c}\dfrac{[\circledcirc(\vec{A})]^a \quad X_1\ldots X_{l-1} \quad \begin{array}{c}\mathfrak{D}_2\\ A_l\end{array} \quad \begin{array}{c}\mathfrak{D}_3\\ \neg A_l\end{array} \quad X_{l+1}\ldots X_n}{B}\\ \mathfrak{E}\\ C\end{array} \quad Y_1\ldots Y_{l-1} \quad \begin{array}{c}[\neg A_l]^e\\ \mathfrak{D}_8\\ C\end{array} \quad Y_{l+1}\ldots Y_n}{C}$$

The reduction procedure is as follows.

$$
\mathfrak{R}_2^{a,d,e}\,\dfrac{\begin{array}{c}\mathfrak{D}_3\\ \neg A_l\\ \mathfrak{D}_8\\ C\\ \mathfrak{E}^*\\ C\end{array}\qquad Y_1 \dots Y_{l-1}\qquad \begin{array}{c}[\neg A_l]^e\\ \mathfrak{D}_8\\ C\end{array}\qquad Y_{l+1}\dots Y_n}{C}
$$

where $\mathfrak{E}^*$ is defined according to the ρ-reduction described in the normalisation proof for Segerberg's system on p. 55.

Let us emphasize that $\circledcirc$ can be a unary connective. But due to our definition of the degree of a formula, the degree of $\circledcirc A$ is higher than the degree of $\neg A$. The case $\mathcal{C}_6$ is considered similarly.

Subcase 2.2. The maximal formula $\neg\circledcirc(\vec{A})$ produced by applications of the rules $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1\big)$ and $R^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}', \mathfrak{h}', \mathfrak{f}'\rangle {}^1\!/\!_2\big)$. Similarly to the previous subcase: replace $\circledcirc(\vec{A})$ with $\neg\circledcirc(\vec{A})$.

Maximal segments. Case 3. *Subcase 3.1.* The maximal segment with formula $\neg\neg B$ and the applications of the rules $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 0\big)$ (or $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle {}^1\!/\!_2\big)$) and $(\neg\neg E)$ (we abbreviate them as $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively).

$$
\mathfrak{R}_2^{d}\,\dfrac{\mathfrak{R}_1^{a,b,c}\,\dfrac{\mathfrak{X}\qquad \begin{array}{c}[\neg A_i^{\dagger}]^b\\ \mathfrak{D}_2^{\dagger}\\ \neg\neg B\end{array}\qquad \begin{array}{c}\mathfrak{D}_3^{\ddagger}\\ A_j^{\ddagger}\end{array}\qquad \begin{array}{c}\mathfrak{D}_4^{\ddagger}\\ \neg A_j^{\ddagger}\end{array}\qquad \begin{array}{c}[A_k^{*}]^c\\ \mathfrak{D}_5^{*}\\ \neg\neg B\end{array}}{\neg\neg B}\qquad \begin{array}{c}[B]^d\\ \mathfrak{D}_6\\ C\end{array}}{C}
$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$, where $\mathfrak{X}$ is one of these two options depending on the applied rule (either $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 0\big)$ in the left option or $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle {}^1\!/\!_2\big)$ in the right option):

$$
\begin{array}{c}\mathfrak{D}_1\\ \circledcirc(\vec{A})\end{array}\qquad\qquad \begin{array}{c}[\circledcirc(\vec{A})]^a\\ \mathfrak{D}_1'\\ \neg\neg B\end{array}
$$

The permutative reduction procedure is as follows.

$$
\mathfrak{R}_1^{a,b,c}\,\dfrac{\mathfrak{Y}\qquad \mathfrak{R}_2^{d}\,\dfrac{\begin{array}{c}[\neg A_i^{\dagger}]^b\\ \mathfrak{D}_2^{\dagger}\\ \neg\neg B\end{array}\quad \begin{array}{c}[B]^d\\ \mathfrak{D}_6\\ C\end{array}}{C}\qquad \begin{array}{c}\mathfrak{D}_3^{\ddagger}\\ A_j^{\ddagger}\end{array}\qquad \begin{array}{c}\mathfrak{D}_4^{\ddagger}\\ \neg A_j^{\ddagger}\end{array}\qquad \mathfrak{R}_2^{d}\,\dfrac{\begin{array}{c}[A_k^{*}]^c\\ \mathfrak{D}_5^{*}\\ \neg\neg B\end{array}\quad \begin{array}{c}[B]^d\\ \mathfrak{D}_6\\ C\end{array}}{C}}{C}
$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$, where $\mathfrak{Y}$ is one of these two options depending on the applied rule (either $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 0\big)$ in the left option or $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle {}^1\!/\!_2\big)$ in the right option):

$$
\begin{array}{c}\mathfrak{D}_1\\ \circledcirc(\vec{A})\end{array}\qquad\qquad \mathfrak{R}_2^{d}\,\dfrac{\begin{array}{c}[\circledcirc(\vec{A})]^a\\ \mathfrak{D}_1'\\ \neg\neg B\end{array}\quad \begin{array}{c}[B]^d\\ \mathfrak{D}_6\\ C\end{array}}{C}
$$

Notice that if C is also on a maximal segment, the procedure increases its length by one. But that can be handled by choosing a suitable maximal segment, i.e., ensuring that the one which have been shortened by the procedure, consists of formulas of a higher degree than C.

Subcase 3.2. The maximal segment with the formula $\neg\neg B$ and the applications of the rules $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1\big)$ (or $R^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle {}^1\!/\!_2\big)$) and $(\neg\neg E)$. Similarly to *Subcase 3.1.*

Case 4. *Subcase 4.1.* The maximal segment with the formula $\circledcirc_1(\vec{B})$ ($\circledcirc_1$ and $\circledcirc_2$ can be distinct operators, but can coincide) and the applications of the rules $R_{\circledcirc_2}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ and $R_{\circledcirc_1}\big(\langle \mathfrak{t}',\mathfrak{h}',\mathfrak{f}'\rangle 0\big)$ (or $R_{\circledcirc_2}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$) which we denote by $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively.

$$\mathfrak{R}_2^{d,e}\;\dfrac{\mathfrak{R}_1^{a,b,c}\;\dfrac{\mathfrak{X}\quad \begin{array}{c}[\neg A_i^{\dagger}]^b\\ \mathfrak{D}_2^{\dagger}\\ \circledcirc_1(\vec{B})\end{array}\quad \begin{array}{c}\mathfrak{D}_3^{\ddagger}\\ A_j^{\ddagger}\end{array}\quad \begin{array}{c}\mathfrak{D}_4^{\ddagger}\\ \neg A_j^{\ddagger}\end{array}\quad \begin{array}{c}[A_k^{*}]^c\\ \mathfrak{D}_5^{*}\\ \circledcirc_1(\vec{B})\end{array}}{\circledcirc_1(\vec{B})}\quad \begin{array}{c}[\neg B_{i'}^{\sharp}]^d\\ \mathfrak{D}_6^{\sharp}\\ C\end{array}\quad \begin{array}{c}\mathfrak{D}_7^{\S}\\ B_{j'}^{\S}\end{array}\quad \begin{array}{c}\mathfrak{D}_8^{\S}\\ \neg B_{j'}^{\S}\end{array}\quad \begin{array}{c}[B_{k'}^{\star}]^e\\ \mathfrak{D}_9^{\star}\\ C\end{array}}{C}$$

†for each $i\in\mathfrak{t}$, ‡for each $j\in\mathfrak{h}$, *for each $k\in\mathfrak{f}$; $^\sharp$for each $i'\in\mathfrak{t}'$, §for each $j'\in\mathfrak{h}'$, *for each $k'\in\mathfrak{f}'$, where $\mathfrak{X}$ is one of the following options depending on the applied rule ($R_{\circledcirc_2}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ in the option on the left side and $R_{\circledcirc_2}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ in the option on the right side):

$$\begin{array}{c}[\circledcirc_2(\vec{A})]^a\\ \mathfrak{D}_1\\ \circledcirc_1(\vec{B})\end{array}\qquad\qquad \begin{array}{c}\mathfrak{D}_1'\\ \circledcirc_2(\vec{A})\end{array}$$

Let us introduce an abbreviation $\mathfrak{A}_2$ for the following derivations:

$$\begin{array}{c}[\neg B_{i'}^{\sharp}]^d\\ \mathfrak{D}_6^{\sharp}\\ C\end{array}\qquad \begin{array}{c}\mathfrak{D}_7^{\S}\\ B_{j'}^{\S}\end{array}\qquad \begin{array}{c}\mathfrak{D}_8^{\S}\\ \neg B_{j'}^{\S}\end{array}\qquad \begin{array}{c}[B_{k'}^{\star}]^e\\ \mathfrak{D}_9^{\star}\\ C\end{array}$$

$^\sharp$for each $i'\in\mathfrak{t}'$, §for each $j'\in\mathfrak{h}'$, *for each $k'\in\mathfrak{f}'$.

The permutative reduction procedure is as follows.

$$\mathfrak{R}_1^{a,b,c}\;\dfrac{\mathfrak{Y}\quad \mathfrak{R}_2^{d,e}\;\dfrac{\begin{array}{c}[\neg A_i^{\dagger}]^b\\ \mathfrak{D}_2^{\dagger}\\ \circledcirc_1(\vec{B})\end{array}\quad \mathfrak{A}_2}{C}\quad \begin{array}{c}\mathfrak{D}_3\\ A_j^{\ddagger}\end{array}\quad \begin{array}{c}\mathfrak{D}_4\\ \neg A_j^{\ddagger}\end{array}\quad \mathfrak{R}_2^{d,e}\;\dfrac{\begin{array}{c}[A_k^{*}]^c\\ \mathfrak{D}_5^{*}\\ \circledcirc_1(\vec{B})\end{array}\quad \mathfrak{A}_2}{C}}{C}$$

†for each $i\in\mathfrak{t}$, ‡for each $j\in\mathfrak{h}$, *for each $k\in\mathfrak{f}$, where $\mathfrak{Y}$ is one of the following options depending on the applied rule ($R_{\circledcirc_2}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ in the option on the left side and $R_{\circledcirc_2}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ in the option on the right side):

$$\mathfrak{R}_2^{d,e}\;\dfrac{\begin{array}{c}[\circledcirc_2(\vec{A})]^a\\ \mathfrak{D}_1\\ \circledcirc_1(\vec{B})\end{array}\quad \mathfrak{A}_2}{C}\qquad\qquad \begin{array}{c}\mathfrak{D}_1'\\ \circledcirc_2(\vec{A})\end{array}$$

Subcase 4.2. The maximal segment with the formula of the form $\neg\circledcirc_1(\vec{B})$ and the applications of the rules $R_{\circledcirc_2}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ (or $R_{\circledcirc_2}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$) and $R_{\circledcirc_1}\big(\langle \mathfrak{t}',\mathfrak{h}',\mathfrak{f}'\rangle 1\big)$. Similar to Subcase 4.1: replace $\circledcirc_1(\vec{B})$ with $\neg\circledcirc_1(\vec{B})$.

Subcase 4.3. The maximal segment with the formula $\circledcirc_1(\vec{B})$ and the applications of the rules $R^{\neg}_{\circledcirc_2}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ (or $R^{\neg}_{\circledcirc_2}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$) and $R_{\circledcirc_1}\big(\langle \mathfrak{t}',\mathfrak{h}',\mathfrak{f}'\rangle 0\big)$. Similar to Subcase 4.1: replace $\circledcirc_2(\vec{A})$ with $\neg\circledcirc_2(\vec{A})$.

Subcase 4.4. The maximal segment with the formula $\neg\circledcirc_1(\vec{B})$ and the applications of the rules $R^{\neg}_{\circledcirc_2}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ (or $R^{\neg}_{\circledcirc_2}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$) and $R_{\circledcirc_1}\big(\langle \mathfrak{t}',\mathfrak{h}',\mathfrak{f}'\rangle 1\big)$. Similar to Subcase 4.1: replace $\circledcirc_2(\vec{A})$ with $\neg\circledcirc_2(\vec{A})$ and $\circledcirc_1(\vec{B})$ with $\neg\circledcirc_1(\vec{B})$.

Case 7. *Subcase 7.1.* Consider the maximal segment with the formula $\circledcirc(\vec{A})$ and the applications of the rules (EM) and $R_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ (we abbreviate them as $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively).

$$
\mathfrak{R}_2^{c,d}\ \dfrac{\mathfrak{R}_1^{a,b}\ \dfrac{\begin{array}{c}[C]^a\\ \mathfrak{D}_1\\ \circledcirc(\vec{A})\end{array}\qquad \begin{array}{c}[\neg C]^b\\ \mathfrak{D}_2\\ \circledcirc(\vec{A})\end{array}}{\circledcirc(\vec{A})}\qquad \begin{array}{c}[\neg A_i^{\dagger}]^c\\ \mathfrak{D}_3^{\dagger}\\ B\end{array}\qquad \begin{array}{c}\mathfrak{D}_4^{\ddagger}\\ A_j^{\ddagger}\end{array}\qquad \begin{array}{c}\mathfrak{D}_5^{\ddagger}\\ \neg A_j^{\ddagger}\end{array}\qquad \begin{array}{c}[A_k^{*}]^d\\ \mathfrak{D}_6^{*}\\ B\end{array}}{B}
$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$.

Let us introduce an abbreviation $\mathfrak{A}_1$ for the following derivations.

$$
\begin{array}{c}[\neg A_i^{\dagger}]^c\\ \mathfrak{D}_3^{\dagger}\\ B\end{array}\qquad \begin{array}{c}\mathfrak{D}_4^{\ddagger}\\ A_j^{\ddagger}\end{array}\qquad \begin{array}{c}\mathfrak{D}_5^{\ddagger}\\ \neg A_j^{\ddagger}\end{array}\qquad \begin{array}{c}[A_k^{*}]^d\\ \mathfrak{D}_6^{*}\\ B\end{array}
$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$.

The permutative reduction procedure shortening the maximal segment by one is as follows.

$$
\mathfrak{R}_1^{a,b}\ \dfrac{\mathfrak{R}_2^{c,d}\ \dfrac{\begin{array}{c}[C]^a\\ \mathfrak{D}_1\\ \circledcirc(\vec{A})\end{array}\qquad \mathfrak{A}_1}{B}\qquad \mathfrak{R}_2^{c,d}\ \dfrac{\begin{array}{c}[\neg C]^b\\ \mathfrak{D}_2\\ \circledcirc(\vec{A})\end{array}\qquad \mathfrak{A}_1}{B}}{B}
$$

Subcase 7.2. The maximal segment with the formula $\neg\circledcirc(\vec{A})$ and the applications of the rules (EM) and $R_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$. It is similar to Subcase 7.1: just replace $\circledcirc(\vec{A})$ with $\neg\circledcirc(\vec{A})$.

3.4.2 The case of $\mathbf{DG_3}$

The negation fragment of $\mathbf{DG_3}$

Maximal formulas. Case 1. The maximal formula of the form $\neg\neg A$ is produced by applications of (EM) and (EFQ$_{\neg}$) (we abbreviate them as $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively). We transform the derivation as follows.

$$
\mathfrak{R}_1^{a,b}\ \dfrac{\begin{array}{c}[\neg A]^a\\ \mathfrak{D}_1\\ C\end{array}\qquad \begin{array}{c}\mathfrak{R}_2\ \dfrac{[\neg\neg A]^b\qquad \begin{array}{c}\mathfrak{D}_2\\ \neg A\end{array}}{B}\\ \mathfrak{E}\\ C\end{array}}{C}
\quad \dashrightarrow \quad
\begin{array}{c}\mathfrak{D}_2\\ \neg A\\ \mathfrak{D}_1\\ C\\ \mathfrak{E}^{*}\\ C\end{array}
$$

where $\mathfrak{E}^{*}$ is defined according to the ρ-reduction described in the normalisation proof for Segerberg's system on p. 55. Due to Lemma 98 there is exactly one formula in the assumption class b (Lemma 98 has to be applied before doing this reduction).

Maximal segments. Case 2. The maximal segment with the formula $\neg\neg B$ is produced by two applications of the rule ($\mathrm{EFQ}_\neg$).

$$\dfrac{\dfrac{\begin{matrix}\mathfrak{D}_1\\ \neg\neg A\end{matrix}\qquad\begin{matrix}\mathfrak{D}_2\\ \neg A\end{matrix}}{\neg\neg B}\qquad\begin{matrix}\mathfrak{D}_3\\ \neg B\end{matrix}}{C}\quad\dashrightarrow\quad\dfrac{\begin{matrix}\mathfrak{D}_1\\ \neg\neg A\end{matrix}\qquad\begin{matrix}\mathfrak{D}_2\\ \neg A\end{matrix}}{C}$$

Case 3. The maximal segment with the formula $\neg\neg A$ and the applications of (EM) and ($\mathrm{EFQ}_\neg$). We transform the derivation as follows.

$$\mathfrak{R}_2\,\dfrac{\mathfrak{R}_1^{a,b}\,\dfrac{\begin{matrix}[B]^a\\ \mathfrak{D}_1\\ \neg\neg A\end{matrix}\qquad\begin{matrix}[\neg B]^b\\ \mathfrak{D}_2\\ \neg\neg A\end{matrix}}{\neg\neg A}\qquad\begin{matrix}\mathfrak{D}_3\\ \neg A\end{matrix}}{C}\quad\dashrightarrow\quad\mathfrak{R}_1^{a,b}\,\dfrac{\mathfrak{R}_2\,\dfrac{\begin{matrix}[B]^a\\ \mathfrak{D}_1\\ \neg\neg A\end{matrix}\qquad\begin{matrix}\mathfrak{D}_3\\ \neg A\end{matrix}}{C}\qquad\mathfrak{R}_2\,\dfrac{\begin{matrix}[\neg B]^b\\ \mathfrak{D}_2\\ \neg\neg A\end{matrix}\qquad\begin{matrix}\mathfrak{D}_3\\ \neg A\end{matrix}}{C}}{C}$$

The negation fragment of $\mathbf{DG_3}$ extended by n-ary operators $\circledcirc_1,\dots,\circledcirc_m$

Since $\mathbf{DG}_{\mathbf{3}\neg}^{(\circledcirc)_m}$ has the same rules for n-ary operators as $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$, the cases when in $\mathbf{DG}_{\mathbf{3}\neg}^{(\circledcirc)_m}$ maximal formulas or maximal segments are produced by these rules, are the same as in $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$. Moreover, both $\mathbf{DG}_{\mathbf{3}\neg}^{(\circledcirc)_m}$ and $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$ have the rule (EM). So, the cases when in $\mathbf{DG}_{\mathbf{3}\neg}^{(\circledcirc)_m}$ maximal formulas or maximal segments are produced by (EM) and the rules for n-ary operators, are the same as in $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$. We need to consider the cases when in $\mathbf{DG}_{\mathbf{3}\neg}^{(\circledcirc)_m}$ maximal formulas or maximal segments are produced by ($\mathrm{EFQ}_\neg$) and the rules for n-ary operators.

Maximal formulas. Case 1. *Subcase 1.1.* The maximal formula $\circledcirc(\vec{A})$ produced by applications of the rules ($\mathrm{EFQ}_\neg$) and $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ (we abbreviate them as $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively).

$$\mathfrak{R}_2^{a,b}\,\dfrac{\mathfrak{R}_1\,\dfrac{\begin{matrix}\mathfrak{D}_1\\ \neg\neg C\end{matrix}\qquad\begin{matrix}\mathfrak{D}_2\\ \neg C\end{matrix}}{\circledcirc(\vec{A})}\qquad\begin{matrix}[\neg A_i^\dagger]^a\\ \mathfrak{D}_3^\dagger\\ B\end{matrix}\qquad\begin{matrix}\mathfrak{D}_4^\ddagger\\ A_j^\ddagger\end{matrix}\qquad\begin{matrix}\mathfrak{D}_5^\ddagger\\ \neg A_j^\ddagger\end{matrix}\qquad\begin{matrix}[A_k^*]^b\\ \mathfrak{D}_6^*\\ B\end{matrix}}{B}\quad\dashrightarrow\quad\mathfrak{R}_1\,\dfrac{\begin{matrix}\mathfrak{D}_1\\ \neg\neg C\end{matrix}\qquad\begin{matrix}\mathfrak{D}_2\\ \neg C\end{matrix}}{B}$$

†for each $i\in\mathfrak{t}$, ‡for each $j\in\mathfrak{h}$, *for each $k\in\mathfrak{f}$.

Subcase 1.2. The maximal formula $\neg\circledcirc(\vec{A})$ produced by applications of the rules ($\mathrm{EFQ}_\neg$) and $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$ (we abbreviate them as $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively). Similarly to the previous case.

Maximal segments. Case 2. *Subcase 2.1.* The maximal segment with the formula $\neg\neg B$ and the applications of the rules $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle{}^1\!/_2\big)$ (or $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$) and ($\mathrm{EFQ}_\neg$) (we abbreviate them as $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively).

$$\mathfrak{R}_2\,\dfrac{\mathfrak{R}_1^{a,b,c}\,\dfrac{\mathfrak{X}\qquad\begin{matrix}[\neg A_i^\dagger]^b\\ \mathfrak{D}_2^\dagger\\ \neg\neg B\end{matrix}\qquad\begin{matrix}\mathfrak{D}_3^\ddagger\\ A_j^\ddagger\end{matrix}\qquad\begin{matrix}\mathfrak{D}_4^\ddagger\\ \neg A_j^\ddagger\end{matrix}\qquad\begin{matrix}[A_k^*]^c\\ \mathfrak{D}_5^*\\ \neg\neg B\end{matrix}}{\neg\neg B}\qquad\begin{matrix}\mathfrak{D}_6\\ \neg B\end{matrix}}{C}$$

†for each $i\in\mathfrak{t}$, ‡for each $j\in\mathfrak{h}$, *for each $k\in\mathfrak{f}$, where $\mathfrak{X}$ is one of these options depending on which rules is applied ($R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle{}^1\!/_2\big)$ in the option displayed on the left side and $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ in the option displayed on the right side):

$$\begin{array}{c}[\circledcirc(\vec{A})]^a\\ \mathfrak{D}_1\\ \neg\neg B\end{array} \qquad \begin{array}{c}\mathfrak{D}_1'\\ \circledcirc(\vec{A})\end{array}$$

The permutative reduction procedure is as follows.

$$\mathfrak{R}_1^{a,b,c}\ \dfrac{\mathfrak{Y} \quad \dfrac{\begin{array}{c}[\neg A_i^\dagger]^b\\ \mathfrak{D}_2^\dagger\\ \neg\neg B\end{array} \quad \begin{array}{c}\mathfrak{D}_6\\ \neg B\end{array}}{C} \quad \begin{array}{c}\mathfrak{D}_3^\ddagger\\ A_j^\ddagger\end{array} \quad \begin{array}{c}\mathfrak{D}_4^\ddagger\\ \neg A_j^\ddagger\end{array} \quad \dfrac{\begin{array}{c}[A_k^*]^c\\ \mathfrak{D}_6^*\\ \neg\neg B\end{array} \quad \begin{array}{c}\mathfrak{D}_5\\ \neg B\end{array}}{C}}{C}$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$, where $\mathfrak{Y}$ is one of these options depending on which rules is applied ($R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle{}^1\!/\!_2\big)$ in the option displayed on the left side and $R_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ in the option displayed on the right side):

$$\mathfrak{R}_2\ \dfrac{\begin{array}{c}[\circledcirc(\vec{A})]^a\\ \mathfrak{D}_1\\ \neg\neg B\end{array} \quad \begin{array}{c}\mathfrak{D}_6\\ \neg B\end{array}}{C} \qquad \begin{array}{c}\mathfrak{D}_1'\\ \circledcirc(\vec{A})\end{array}$$

Subcase 2.2. The maximal segment with the formula $\neg\neg B$ and the applications of the rules $R_\circledcirc^\neg\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle{}^1\!/\!_2\big)$ and (EFQ$_\neg$). Similar to Subcase 2.1: replace $\circledcirc(\vec{A})$ with $\neg\circledcirc(\vec{A})$.

3.4.3 The case of $\mathbf{K_3}$

The negation fragment of $\mathbf{K_3}$

Maximal formulas. Case 1. The maximal formula of the form $\neg\neg A$ is produced by applications of the rules $(\neg\neg I)$ and $(\neg\neg E)$. This case coincides with Case 1 from the proof of the normalisation theorem for the negation fragment of **LP**.

Case 2. The maximal formula $\neg\neg A$ is produced by two applications of the rules (EFQ) and $(\neg\neg E)$. We transform the derivation as follows.

$$(\neg\neg E)^a\ \dfrac{(\text{EFQ})\ \dfrac{\begin{array}{c}\mathfrak{D}_1\\ \neg B\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ B\end{array}}{\neg\neg A} \quad \begin{array}{c}[A]^a\\ \mathfrak{D}_3\\ C\end{array}}{C} \quad \dashrightarrow \quad (\text{EFQ})\ \dfrac{\begin{array}{c}\mathfrak{D}_1\\ \neg B\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ B\end{array}}{C}$$

Maximal segments. Case 3. The maximal segment $\neg A$ is produced by two applications of the rule (EFQ). We transform the derivation as follows.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \neg B\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ B\end{array}}{\neg A} \quad \begin{array}{c}\mathfrak{D}_3\\ A\end{array}}{C} \quad \dashrightarrow \quad \dfrac{\begin{array}{c}\mathfrak{D}_1\\ \neg B\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ B\end{array}}{C}$$

Case 4. The maximal segment with the formula $\neg\neg A$ is produced by applications of the rules $(\neg\neg I)$ and (EFQ). We transform the derivation as follows.

$$(\text{EFQ})\ \dfrac{(\neg\neg I)^a\ \dfrac{\begin{array}{c}\mathfrak{D}_1\\ C\end{array} \quad \begin{array}{c}[\neg\neg C]^a\\ \mathfrak{D}_2\\ \neg\neg A\end{array}}{\neg\neg A} \quad \begin{array}{c}\mathfrak{D}_3\\ \neg A\end{array}}{B} \quad \dashrightarrow \quad (\neg\neg I)^a\ \dfrac{\begin{array}{c}\mathfrak{D}_1\\ C\end{array} \quad (\text{EFQ})\ \dfrac{\begin{array}{c}[\neg\neg C]^a\\ \mathfrak{D}_2\\ \neg\neg A\end{array} \quad \begin{array}{c}\mathfrak{D}_3\\ \neg A\end{array}}{B}}{B}$$

The negation fragment of $\mathbf{K_3}$ extended by n-ary operators $\circledcirc_1, \ldots, \circledcirc_m$

Maximal formulas. Case 1. *Subcase 1.1.* The maximal formula $\circledcirc(\vec{A})$ produced by applications of the rules $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1/2\big)$ and $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}', \mathfrak{h}', \mathfrak{f}'\rangle 1\big)$ which we denote by $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively.

$$
\mathfrak{R}_2^{a,d,e}\,\dfrac{\mathfrak{R}_1^{b,c}\,\dfrac{[\circledcirc(\vec{A})]^a \quad \begin{array}{c}\mathfrak{D}_1^\dagger\\ A_i^\dagger\end{array} \quad \begin{array}{c}[A_j^\ddagger]^b\\ \mathfrak{D}_2^\ddagger\\ B\end{array} \quad \begin{array}{c}[\neg A_j^\ddagger]^c\\ \mathfrak{D}_3^\ddagger\\ B\end{array} \quad \begin{array}{c}\mathfrak{D}_4^*\\ \neg A_k^*\end{array}}{\begin{array}{c}B\\ \mathfrak{E}\\ C\end{array}} \quad \begin{array}{c}\mathfrak{D}_5^\sharp\\ A_{i'}^\sharp\end{array} \quad \begin{array}{c}[A_{j'}^\S]^d\\ \mathfrak{D}_6^\S\\ C\end{array} \quad \begin{array}{c}[\neg A_{j'}^\S]^e\\ \mathfrak{D}_7^\S\\ C\end{array} \quad \begin{array}{c}\mathfrak{D}_8^\star\\ \neg A_{k'}^\star\end{array}}{C}
$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$; $^\sharp$for each $i' \in \mathfrak{t}'$, §for each $j' \in \mathfrak{h}'$, *for each $k' \in \mathfrak{f}'$.

Recall that all the formulas $A_i, A_j, \neg A_j, \neg A_k, A_{i'}, A_{j'}, \neg A_{j'}, \neg A_{k'}$ are either subformulas or negations of subformulas of $\circledcirc(\vec{A}) = \circledcirc(A_1, \ldots, A_n)$. This case can be reformulated in the following way.

$$
\mathfrak{R}_2^{a,d,e}\,\dfrac{\mathfrak{R}_1^{b,c}\,\dfrac{[\circledcirc(\vec{A})]^a \quad X_1 \quad \ldots \quad X_n}{\begin{array}{c}B\\ \mathfrak{E}\\ C\end{array}} \quad Y_1 \quad \ldots \quad Y_n}{C}
$$

where $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are abbreviations for the following derivations, for any $u, t \in \{1, \ldots, n\}$:

$$
X_u = \left\{\begin{array}{lll}
\begin{array}{c}\mathfrak{D}_1\\ A_u\end{array} & \textit{iff} & u \in \mathfrak{f};\\[2ex]
\begin{array}{cc}[A_u]^b & [\neg A_u]^c\\ \mathfrak{D}_2 & \mathfrak{D}_3\\ B & B\end{array} & \textit{iff} & u \in \mathfrak{h};\\[2ex]
\begin{array}{c}\mathfrak{D}_4\\ \neg A_u\end{array} & \textit{iff} & u \in \mathfrak{t};
\end{array}\right.
\qquad
Y_t = \left\{\begin{array}{lll}
\begin{array}{c}\mathfrak{D}_5\\ A_t\end{array} & \textit{iff} & t \in \mathfrak{f}';\\[2ex]
\begin{array}{cc}[A_t]^d & [\neg A_t]^e\\ \mathfrak{D}_6 & \mathfrak{D}_7\\ C & C\end{array} & \textit{iff} & t \in \mathfrak{h}';\\[2ex]
\begin{array}{c}\mathfrak{D}_8\\ \neg A_t\end{array} & \textit{iff} & t \in \mathfrak{t}'.
\end{array}\right.
$$

As follows from these equalities and soundness of our natural deduction systems, there is $l \in \{1, \ldots, n\}$ such that $X_l \neq Y_l$. Then following combinations are possible (we present them in the form of ordered pairs $\langle X_l, Y_l\rangle$):

$$
\mathcal{C}_1 = \left\langle \begin{array}{c}\mathfrak{D}_1\\ A_l\end{array}, \begin{array}{cc}[A_l]^d & [\neg A_l]^e\\ \mathfrak{D}_6 & \mathfrak{D}_7\\ C & C\end{array} \right\rangle \quad
\mathcal{C}_2 = \left\langle \begin{array}{c}\mathfrak{D}_1\\ A_l\end{array}, \begin{array}{c}\mathfrak{D}_8\\ \neg A_l\end{array} \right\rangle
$$

$$
\mathcal{C}_3 = \left\langle \begin{array}{cc}[A_l]^b & [\neg A_l]^c\\ \mathfrak{D}_2 & \mathfrak{D}_3\\ B & B\end{array}, \begin{array}{c}\mathfrak{D}_5\\ A_l\end{array} \right\rangle \quad
\mathcal{C}_4 = \left\langle \begin{array}{cc}[A_l]^b & [\neg A_l]^c\\ \mathfrak{D}_2 & \mathfrak{D}_3\\ B & B\end{array}, \begin{array}{c}\mathfrak{D}_8\\ \neg A_l\end{array} \right\rangle
$$

$$
\mathcal{C}_5 = \left\langle \begin{array}{c}\mathfrak{D}_4\\ \neg A_l\end{array}, \begin{array}{c}\mathfrak{D}_5\\ A_l\end{array} \right\rangle \quad
\mathcal{C}_6 = \left\langle \begin{array}{c}\mathfrak{D}_4\\ \neg A_l\end{array}, \begin{array}{cc}[A_l]^d & [\neg A_l]^e\\ \mathfrak{D}_6 & \mathfrak{D}_7\\ C & C\end{array} \right\rangle
$$

Let us consider the case $\mathcal{C}_2$.

$$
\mathfrak{R}_2^{a,d,e}\,\dfrac{\begin{array}{c}\mathfrak{R}_1^{b,c}\,\dfrac{[\circledcirc(\vec{A})]^a \quad X_1\ldots X_{l-1} \quad \begin{array}{c}\mathfrak{D}_1\\ A_l\end{array} \quad X_{l+1}\ldots X_n}{B}\\ \mathfrak{E}\\ C\end{array} \quad Y_1\ldots Y_{l-1} \quad \begin{array}{c}\mathfrak{D}_8\\ \neg A_l\end{array} \quad Y_{l+1}\ldots Y_n}{C}
$$

The reduction procedure is as follows:

$$
\mathfrak{R}_2^{a,d,e}\,\dfrac{\begin{array}{c}(\mathrm{EFQ})\,\dfrac{\begin{array}{c}\mathfrak{D}_8\\ \neg A_l\end{array} \quad \begin{array}{c}\mathfrak{D}_1\\ A_l\end{array}}{B}\\ \mathfrak{E}\\ C\end{array} \quad Y_1\ldots Y_{l-1} \quad \begin{array}{c}\mathfrak{D}_8\\ \neg A_l\end{array} \quad Y_{l+1}\ldots Y_n}{C}
$$

The other cases are considered similarly.

Subcase 1.2. The maximal formula $\neg\circledcirc(\vec{A})$ produced by applications of the rules $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ and $\widetilde{R}^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}',\mathfrak{h}',\mathfrak{f}'\rangle 0\big)$. Similarly to the previous subcase: replace $\circledcirc(\vec{A})$ with $\neg\circledcirc(\vec{A})$.

Case 2. *Subcase 2.1.* The maximal formula $\circledcirc(\vec{A})$ produced by applications of the rules (EFQ) and $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ which we denote by $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively.

$$
\mathfrak{R}_2^{a,b}\,\dfrac{\mathfrak{R}_1\,\dfrac{\begin{array}{c}\mathfrak{D}_1\\ C\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ \neg C\end{array}}{\circledcirc(\vec{A})} \quad \begin{array}{c}\mathfrak{D}_3^{\dagger}\\ A_i^{\dagger}\end{array} \quad \begin{array}{c}[A_j^{\ddagger}]^a\\ \mathfrak{D}_4^{\ddagger}\\ B\end{array} \quad \begin{array}{c}[\neg A_j^{\ddagger}]^b\\ \mathfrak{D}_5^{\ddagger}\\ B\end{array} \quad \begin{array}{c}\mathfrak{D}_6^{*}\\ \neg A_k^{*}\end{array}}{B} \quad \dashrightarrow \quad \mathfrak{R}_1\,\dfrac{\begin{array}{c}\mathfrak{D}_1\\ C\end{array} \quad \begin{array}{c}\mathfrak{D}_2\\ \neg C\end{array}}{B}
$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$.

Subcase 2.2. The maximal formula $\neg\circledcirc(\vec{A})$ produced by applications of the rules (EFQ) and $\widetilde{R}^{\neg}_{\circledcirc}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$. Similarly to the previous case.

Case 3. *Subcase 3.1.* The maximal segment with the formula $\neg B$ and the applications of the rules $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$ (or $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$) and (EFQ) (we abbreviate them as $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively).

$$
\mathfrak{R}_2\,\dfrac{\mathfrak{R}_1^{a,b,c}\,\dfrac{\mathfrak{X} \quad \begin{array}{c}\mathfrak{D}_2^{\dagger}\\ A_i^{\dagger}\end{array} \quad \begin{array}{c}[A_j^{\ddagger}]^b\\ \mathfrak{D}_3^{\ddagger}\\ \neg B\end{array} \quad \begin{array}{c}[\neg A_j^{\ddagger}]^c\\ \mathfrak{D}_4^{\ddagger}\\ \neg B\end{array} \quad \begin{array}{c}\mathfrak{D}_5^{*}\\ \neg A_k^{*}\end{array}}{\neg B} \quad \begin{array}{c}\mathfrak{D}_6\\ B\end{array}}{C}
$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$, where $\mathfrak{X}$ is one of the following options depending on the applied rule ($\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$ in the option displayed on the left side, $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2\big)$ in the option displayed on the right side):

$$
\begin{array}{c}[\circledcirc(\vec{A})]^a\\ \mathfrak{D}_1\\ \neg B\end{array} \qquad \begin{array}{c}\mathfrak{D}_1'\\ \circledcirc(\vec{A})\end{array}
$$

The permutative reduction procedure is as follows.

$$\mathfrak{R}_1^{a,b,c}\ \dfrac{\mathfrak{Y} \qquad \begin{array}{c}\mathfrak{D}_2^{\dagger}\\ A_i^{\dagger}\end{array} \qquad \dfrac{\begin{array}{c}[A_j^{\ddagger}]^b\\ \mathfrak{D}_3^{\ddagger}\\ \neg B\end{array} \qquad \begin{array}{c}\mathfrak{D}_6\\ B\end{array}}{C} \qquad \dfrac{\begin{array}{c}[\neg A_j^{\ddagger}]^c\\ \mathfrak{D}_4^{\ddagger}\\ \neg B\end{array} \qquad \begin{array}{c}\mathfrak{D}_6\\ B\end{array}}{C} \qquad \begin{array}{c}\mathfrak{D}_5^{*}\\ \neg A_k^{*}\end{array}}{C}$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$, where $\mathfrak{Y}$ is one of the following options depending on the applied rule ($\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1\big)$ in the option displayed on the left side, $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1/2\big)$ in the option displayed on the right side):

$$\mathfrak{R}_2\ \dfrac{\begin{array}{c}[\circledcirc(\vec{A})]^a\\ \mathfrak{D}_1\\ \neg B\end{array} \qquad \begin{array}{c}\mathfrak{D}_6\\ B\end{array}}{C} \qquad\qquad \begin{array}{c}\mathfrak{D}_1'\\ \circledcirc(\vec{A})\end{array}$$

Subcase 3.2. The maximal segment with the formula $\neg B$ and the applications of the rules $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 0\big)$ (or $\widetilde{R}^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1/2\big)$) and (EFQ). Similar to the Subcase 3.1.

Case 4. *Subcase 4.1.* The maximal segment with the formula $\neg\neg B$ and the applications of the rules $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1\big)$ (or $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1/2\big)$) and $(\neg\neg E)$ (we abbreviate them as $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively).

$$\mathfrak{R}_2^{d}\ \dfrac{\mathfrak{R}_1^{a,b,c}\ \dfrac{\mathfrak{X} \qquad \begin{array}{c}\mathfrak{D}_2^{\dagger}\\ A_i^{\dagger}\end{array} \qquad \begin{array}{c}[A_j^{\ddagger}]^b\\ \mathfrak{D}_3^{\ddagger}\\ \neg\neg B\end{array} \qquad \begin{array}{c}[\neg A_j^{\ddagger}]^c\\ \mathfrak{D}_4^{\ddagger}\\ \neg\neg B\end{array} \qquad \begin{array}{c}\mathfrak{D}_5^{*}\\ \neg A_k^{*}\end{array}}{\neg\neg B} \qquad \begin{array}{c}[B]^d\\ \mathfrak{D}_6\\ C\end{array}}{C}$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$, where $\mathfrak{X}$ is one of the following options depending on the applied rule ($\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1\big)$ in the option displayed on the left side, $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1/2\big)$ in the option displayed on the right side):

$$\begin{array}{c}[\circledcirc(\vec{A})]^a\\ \mathfrak{D}_1\\ \neg\neg B\end{array} \qquad\qquad \begin{array}{c}\mathfrak{D}_1'\\ \circledcirc(\vec{A})\end{array}$$

The permutative reduction procedure is as follows.

$$\mathfrak{R}_1^{a,b,c}\ \dfrac{\mathfrak{Y} \qquad \begin{array}{c}\mathfrak{D}_2^{\dagger}\\ A_i^{\dagger}\end{array} \qquad \mathfrak{R}_2^{d}\ \dfrac{\begin{array}{c}[A_j^{\ddagger}]^b\\ \mathfrak{D}_3^{\ddagger}\\ \neg\neg B\end{array} \qquad \begin{array}{c}[B]^d\\ \mathfrak{D}_6\\ C\end{array}}{C} \qquad \mathfrak{R}_2^{d}\ \dfrac{\begin{array}{c}[\neg A_j^{\ddagger}]^c\\ \mathfrak{D}_4^{\ddagger}\\ \neg\neg B\end{array} \qquad \begin{array}{c}[B]^d\\ \mathfrak{D}_6\\ C\end{array}}{C} \qquad \begin{array}{c}\mathfrak{D}_5^{*}\\ \neg A_k^{*}\end{array}}{B}$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$, where $\mathfrak{Y}$ is one of the following options depending on the applied rule ($\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1\big)$ in the option displayed on the left side, $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1/2\big)$ in the option displayed on the right side):

$$\mathfrak{R}_2^{d}\ \dfrac{\begin{array}{c}[\circledcirc(\vec{A})]^a\\ \mathfrak{D}_1\\ \neg\neg B\end{array} \qquad \begin{array}{c}[B]^d\\ \mathfrak{D}_6\\ C\end{array}}{C} \qquad\qquad \begin{array}{c}\mathfrak{D}_1'\\ \circledcirc(\vec{A})\end{array}$$

Subcase 4.2. The maximal segment with the formula $\neg\neg B$ and the applications of the rules $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 0\big)$ (or $\widetilde{R}^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1/2\big)$) and $(\neg\neg E)$. Similarly to *subcase 4.1.*

Case 5. *Subcase 5.1.* The maximal segment with the formula $\circledcirc_1(\vec{B})$ and the applications of the rules $\widetilde{R}_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$ (or $\widetilde{R}_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle{}^1\!/\!_2\big)$) and $\widetilde{R}_{\circledcirc_1}\big(\langle\mathfrak{t}',\mathfrak{h}',\mathfrak{f}'\rangle{}^1\!/\!_2\big)$ which we denote by $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively.

$$
\mathfrak{R}_2^{d,e}\,\dfrac{
\mathfrak{R}_1^{a,b,c}\,\dfrac{
\begin{array}{c}[\circledcirc_2(\vec{A})]^a\\ \mathfrak{D}_1\\ \circledcirc_1(\vec{B})\end{array}
\quad
\begin{array}{c}\mathfrak{D}_2^\dagger\\ A_i^\dagger\end{array}
\quad
\begin{array}{c}[A_j^\ddagger]^b\\ \mathfrak{D}_3^\ddagger\\ \circledcirc_1(\vec{B})\end{array}
\quad
\begin{array}{c}[\neg A_j^\ddagger]^c\\ \mathfrak{D}_4^\ddagger\\ \circledcirc_1(\vec{B})\end{array}
\quad
\begin{array}{c}\mathfrak{D}_5^*\\ \neg A_k^*\end{array}
}{\circledcirc_1(\vec{B})}
\quad
\begin{array}{c}\mathfrak{D}_6^\sharp\\ B_{i'}^\sharp\end{array}
\quad
\begin{array}{c}[B_{j'}^\S]^d\\ \mathfrak{D}_7^\S\\ C\end{array}
\quad
\begin{array}{c}[\neg B_{j'}^\S]^e\\ \mathfrak{D}_8^\S\\ C\end{array}
\quad
\begin{array}{c}\mathfrak{D}_9^\star\\ \neg B_{k'}^\star\end{array}
}{C}
$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$; $^\sharp$for each $i' \in \mathfrak{t}'$, §for each $j' \in \mathfrak{h}'$, *for each $k' \in \mathfrak{f}'$, where $\mathfrak{X}$ is one of the following options depending on the applied rule ($\widetilde{R}_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$ in the option displayed on the left side, $\widetilde{R}_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle{}^1\!/\!_2\big)$ in the option displayed on the right side):

$$
\begin{array}{c}[\circledcirc_2(\vec{A})]^a\\ \mathfrak{D}_1\\ \circledcirc_1(\vec{B})\end{array}
\qquad\qquad
\begin{array}{c}\mathfrak{D}_1'\\ \circledcirc_2(\vec{A})\end{array}
$$

Let us introduce an abbreviation $\mathfrak{A}_3$ for the following derivations.

$$
\begin{array}{c}\mathfrak{D}_6^\sharp\\ B_{i'}^\sharp\end{array}
\qquad
\begin{array}{c}[B_{j'}^\S]^d\\ \mathfrak{D}_7^\S\\ C\end{array}
\qquad
\begin{array}{c}[\neg B_{j'}^\S]^e\\ \mathfrak{D}_8^\S\\ C\end{array}
\qquad
\begin{array}{c}\mathfrak{D}_9^\star\\ \neg B_{k'}^\star\end{array}
$$

$^\sharp$for each $i' \in \mathfrak{t}'$, §for each $j' \in \mathfrak{h}'$, *for each $k' \in \mathfrak{f}'$.

The permutative reduction procedure is as follows.

$$
\mathfrak{R}_1^{a,b,c}\,\dfrac{
\mathfrak{Y}
\quad
\begin{array}{c}\mathfrak{D}_2^\dagger\\ A_i^\dagger\end{array}
\quad
\mathfrak{R}_2^{d,e}\,\dfrac{\begin{array}{c}[A_j^\ddagger]^b\\ \mathfrak{D}_3^\ddagger\\ \circledcirc(\vec{B})\end{array}\quad \mathfrak{A}_3}{C}
\quad
\mathfrak{R}_2^{d,e}\,\dfrac{\begin{array}{c}[\neg A_j^\ddagger]^c\\ \mathfrak{D}_4^\ddagger\\ \circledcirc(\vec{B})\end{array}\quad \mathfrak{A}_3}{C}
\quad
\begin{array}{c}\mathfrak{D}_5^*\\ \neg A_k^*\end{array}
}{C}
$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$, where $\mathfrak{Y}$ is one of the following options depending on the applied rule ($\widetilde{R}_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$ in the option displayed on the left side, $\widetilde{R}_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle{}^1\!/\!_2\big)$ in the option displayed on the right side):

$$
\mathfrak{R}_2^{d,e}\,\dfrac{\begin{array}{c}[\circledcirc(\vec{A})]^a\\ \mathfrak{D}_1\\ \circledcirc(\vec{B})\end{array}\quad \mathfrak{A}_3}{C}
\qquad\qquad
\begin{array}{c}\mathfrak{D}_1'\\ \circledcirc_2(\vec{A})\end{array}
$$

Subcase 5.2. The maximal segment with the formula $\neg\circledcirc_1(\vec{B})$ and the applications of the rules $\widetilde{R}_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1\big)$ (or $\widetilde{R}_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle{}^1\!/\!_2\big)$) and $\widetilde{R}^{\neg}_{\circledcirc_1}\big(\langle\mathfrak{t}',\mathfrak{h}',\mathfrak{f}'\rangle{}^1\!/\!_2\big)$. Similar to the Subcase 5.1.

Subcase 5.3. The maximal segment with the formula $\circledcirc_1(\vec{B})$ and the applications of the rules $\widetilde{R}_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ (or $\widetilde{R}^{\neg}_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle{}^1\!/\!_2\big)$) and $\widetilde{R}_{\circledcirc_1}\big(\langle\mathfrak{t}',\mathfrak{h}',\mathfrak{f}'\rangle{}^1\!/\!_2\big)$. Similar to the Subcase 5.1.

Subcase 5.4. The maximal segment with the formula $\neg\circledcirc_1(\vec{B})$ and the applications of the rules $\widetilde{R}_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0\big)$ (or $\widetilde{R}^{\neg}_{\circledcirc_2}\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle{}^1\!/\!_2\big)$) and $\widetilde{R}^{\neg}_{\circledcirc_1}\big(\langle\mathfrak{t}',\mathfrak{h}',\mathfrak{f}'\rangle{}^1\!/\!_2\big)$. Similar to the Subcase 5.1.

3.4.4 The case of $\mathbf{G_3}$

Lemma 99. *Any deduction in* $\mathbf{G}_{\mathbf{3}_\neg}^{(\circledcirc)m}$ *can be transformed into one in which every application of the rule* (EM$_\neg$) *discharges exactly one major assumption.*

Proof. Similarly to Lemma 98. □

The negation fragment of $\mathbf{G_3}$

Maximal formulas. Case 1. The maximal formula of the form $\neg\neg A$ is produced by applications of the rules (EM$_\neg$) and (EFQ) (we abbreviate them as $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively). We transform the derivation as follows.

$$\mathfrak{R}_1^{a,b}\dfrac{\begin{matrix}[\neg A]^a\\ \mathfrak{D}_1\\ C\end{matrix} \qquad \begin{matrix}\mathfrak{R}_2\dfrac{[\neg\neg A]^b \qquad \begin{matrix}\mathfrak{D}_2\\ \neg A\end{matrix}}{B}\\ \mathfrak{E}\\ C\end{matrix}}{C} \quad \dashrightarrow \quad \begin{matrix}\mathfrak{D}_2\\ \neg A\\ \mathfrak{D}_1\\ B\\ \mathfrak{E}\\ C\\ \mathfrak{E}^*\\ C\end{matrix}$$

where $\mathfrak{E}^*$ is defined according to the ρ-reduction described in the normalisation proof for Segerberg's system on p. 55. Note that due to Lemma 99 there is exactly one formula in the assumption class b (Lemma 99 has to be applied before doing this reduction).

Maximal segments. Case 2. The maximal segment $\neg A$ is produced by two applications of the rule (EFQ). Similarly for the Case of the proof the negation fragment of $\mathbf{K_3}$.

Case 3. The maximal segment with the formula $\neg A$ and the applications of the rules (EM$_\neg$) and (EFQ) (we abbreviate them as $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively). The permutative reduction procedure is as follows.

$$\mathfrak{R}_2\dfrac{\mathfrak{R}_1^{a,b}\dfrac{\begin{matrix}[\neg B]^a\\ \mathfrak{D}_2\\ \neg A\end{matrix} \qquad \begin{matrix}[\neg\neg B]^b\\ \mathfrak{D}_3\\ \neg A\end{matrix}}{\neg A} \qquad \begin{matrix}\mathfrak{D}_1\\ A\end{matrix}}{C} \quad \dashrightarrow \quad \mathfrak{R}_1^{a,b}\dfrac{\mathfrak{R}_2\dfrac{\begin{matrix}[\neg B]^a\\ \mathfrak{D}_2\\ \neg A\end{matrix} \qquad \begin{matrix}\mathfrak{D}_1\\ A\end{matrix}}{C} \qquad \mathfrak{R}_2\dfrac{\begin{matrix}[\neg\neg B]^b\\ \mathfrak{D}_3\\ \neg A\end{matrix} \qquad \begin{matrix}\mathfrak{D}_1\\ A\end{matrix}}{C}}{C}$$

The negation fragment of $\mathbf{G_3}$ extended by n-ary operators $\circledcirc_1, \ldots, \circledcirc_m$

Since $\mathbf{G}_{\mathbf{3}_\neg}^{(\circledcirc)m}$ has the same rules for n-ary operators as $\mathbf{K}_{\mathbf{3}_\neg}^{(\circledcirc)m}$, the cases when in $\mathbf{G}_{\mathbf{3}_\neg}^{(\circledcirc)m}$ maximal formulas or maximal segments are produced by these rules, are the same as in $\mathbf{K}_{\mathbf{3}_\neg}^{(\circledcirc)m}$. Moreover, both $\mathbf{G}_{\mathbf{3}_\neg}^{(\circledcirc)m}$ and $\mathbf{K}_{\mathbf{3}_\neg}^{(\circledcirc)m}$ have the rule (EFQ). So, the cases when in $\mathbf{G}_{\mathbf{3}_\neg}^{(\circledcirc)m}$ maximal formulas or maximal segments are produced by (EFQ) and the rules for n-ary operators, are the same as in $\mathbf{K}_{\mathbf{3}_\neg}^{(\circledcirc)m}$. We need to consider the cases when in $\mathbf{G}_{\mathbf{3}_\neg}^{(\circledcirc)m}$ maximal segments are produced by (EM$_\neg$) and the rules for n-ary operators.

The maximal segment with the formula $\circledcirc(\vec{A})$ is produced by applications of the rules (EM$_\neg$) and $\widetilde{R}_\circledcirc\big(\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle^{1/2}\big)$ (we use for them the abbreviations $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively).

$$\mathfrak{R}_2^{c,d}\dfrac{\mathfrak{R}_1^{a,b}\dfrac{\begin{matrix}[\neg C]^a\\ \mathfrak{D}_1\\ \circledcirc(\vec{A})\end{matrix} \qquad \begin{matrix}[\neg\neg C]^b\\ \mathfrak{D}_2\\ \circledcirc(\vec{A})\end{matrix}}{\circledcirc(\vec{A})} \qquad \begin{matrix}\mathfrak{D}_3^\dagger\\ A_i^\dagger\end{matrix} \qquad \begin{matrix}[A_j^\ddagger]^c\\ \mathfrak{D}_4^\ddagger\\ B\end{matrix} \qquad \begin{matrix}[\neg A_j^\ddagger]^d\\ \mathfrak{D}_5^\ddagger\\ B\end{matrix} \qquad \begin{matrix}\mathfrak{D}_6^*\\ \neg A_k^*\end{matrix}}{B}$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$.

We introduce an abbreviation $\mathfrak{A}_4$ for the following derivations.

$$\begin{array}{cccc} & [A_j^\ddagger]^c & [\neg A_j^\ddagger]^d & \\ \mathfrak{D}_3^\dagger & \mathfrak{D}_4^\ddagger & \mathfrak{D}_5^\ddagger & \mathfrak{D}_6^* \\ A_i^\dagger & B & B & \neg A_k^* \end{array}$$

†for each $i \in \mathfrak{t}$, ‡for each $j \in \mathfrak{h}$, *for each $k \in \mathfrak{f}$.

Then the permutative reduction procedure is as follows:

$$\mathfrak{R}_1^{a,b}\,\dfrac{\mathfrak{R}_2^{c,d}\,\dfrac{\begin{array}{c}[\neg C]^a\\ \mathfrak{D}_1\\ \circledcirc(\vec{A})\end{array} \quad \mathfrak{A}_4}{B} \qquad \mathfrak{R}_2^{c,d}\,\dfrac{\begin{array}{c}[\neg\neg C]^b\\ \mathfrak{D}_2\\ \circledcirc(\vec{A})\end{array} \quad \mathfrak{A}_4}{B}}{B}$$

The case when the maximal segment with the formula $\neg\circledcirc(\vec{A})$ is produced by applications of the rules ($\mathrm{EM}_\neg$) and $\widetilde{R}^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1/2\big)$ is similar to the previous one.

3.4.5 Theorems

Theorem 100. *Let* $\mathbf{L} \in \{\mathbf{LP}, \mathbf{DG_3}, \mathbf{K_3}, \mathbf{G_3}\}$. *Any deduction in the negation fragment of* $\mathbf{L}$ *or its extensions by* $\circledcirc_1, \ldots, \circledcirc_m$ *can be converted into a deduction in normal form.*

Proof. By induction over the rank of deductions. Similarly to Theorem 56. □

Corollary 101. *If* $\Gamma \vdash A$ *in one of the logics in question, then there is a deduction in normal form with an occurrence of* A *as the conclusion and occurrences of the formulas in* Γ *as the undischarged assumptions.*

Theorem 102. *If* $\mathfrak{D}$ *is a deduction in normal form in one of the logics in question, then all major premises of elimination rules are (discharged or undischarged) assumptions of* $\mathfrak{D}$.

Proof. By the form of deductions in normal form, as a result of the permutative reduction procedures. □

Corollary 103. *If any major premises of elimination rules are on a branch in a deduction in normal form, then they precede any major assumptions discharged by introduction rules that are on the branch.*

Proof. Follows from Theorem 102. □

Theorem 104. *Deductions in normal forms in the negation fragments and their extensions by n-ary connectives* $\circledcirc_1, \ldots, \circledcirc_m$ *in* $\mathbf{LP}$, $\mathbf{K_3}$, $\mathbf{DG_3}$ *and* $\mathbf{G_3}$ *have the negation subformula property.*

Proof. By inspection of the rules and an induction over the order of branches. □

3.5 Sequent calculi for the three-valued logics in question

In this section, we present cut-free sequent calculi for the three-valued logics in question based on the normalised natural deduction systems from the previous sections. As in Section 2.3, we understand a sequent as an ordered pair written as $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas. Obviously, we need an axiom $A \Rightarrow A$[8], (internal) weakening and contraction rules, and the rule of cut.

Now we need to transform natural deduction rules into sequent ones. The rules $(\neg\neg E)$ and $(\neg\neg I)$ can be transformed into the following rules:

$$(\neg\neg\Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta}{\neg\neg A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg\neg)\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg\neg A}$$

Indeed, we can easily prove $(\neg\neg E)$ and $(\neg\neg I)$ via $(\neg\neg\Rightarrow)$, $(\Rightarrow\neg\neg)$, structural rules, and axiom.

$$(\text{Cut})\ \frac{\Gamma \Rightarrow \neg\neg A \qquad (\Rightarrow\neg\neg)\ \dfrac{A \Rightarrow B}{\neg\neg A \Rightarrow B}}{\Gamma \Rightarrow B} \qquad (\text{Cut})\ \frac{(\Rightarrow\neg\neg)\ \dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \neg\neg A} \qquad \neg\neg A \Rightarrow B}{\Gamma \Rightarrow B}$$

If we rewrite $(\neg\neg\Rightarrow)$ and $(\Rightarrow\neg\neg)$ in a natural deduction form requiring Δ be a single formula and Γ be a single formula (at least in the premiss), then we just get just the rules $(\neg\neg I)$ via $(\neg\neg\Rightarrow)$. Of course, these restrictions on Γ and Δ may raise some doubts if we indeed obtain equivalent calculi. However, in the next chapter we will present a proper completeness proof (and cut admissibility as well) for the sequent calculi in question (moreover, supplied with modalities), so we should not worry about that. Our task now is to give a formulation of sequent rules for the logics in question, using their natural deduction formulations.

Let us go to the other rules. For (EM) we have several options: they can be presented as an axiom or as one of the rules given below; due to Indrzejczak's rule generator theorem [82] all these option are equivalent (it can also be easily justified by a Hintikka-style argument in the next chapter: for completeness, it is enough to prove $\Rightarrow A, \neg A$ which is either an axiom or is provable with the help of any rule presented below).

$$(\text{EM}^{ax}) \Rightarrow A, \neg A \qquad (\Rightarrow \text{EM})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A, \neg A} \qquad (\Rightarrow\neg)\ \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$$

$$(\Rightarrow\neg^{-1})\ \frac{\neg A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad (\text{EM}\Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \quad \neg A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

The axiom (EM^{ax}) is used by Avron for the formulation of sequent calculi for **LP** and its extensions [5]. The rule $(\text{EM}\Rightarrow)$ (in a form such that $\Gamma = \Pi$ and $\Delta = \Sigma = B$) is considered by Negri and von Plato [131]. Although all these options are equivalent from the view of completeness and obtained as its consequence cut admissibility, there is a difference, if we are talking about a constructive proof of cut admissibility: at least we are able to present such a proof only if the rule $(\text{EM}\Rightarrow)$ is used.

As for the rule (EFQ), the situation is similar. We have five equivalent options, but from the point of view of a constructive cut elimination, the last option, $(\Rightarrow \text{EFQ})$, is preferable.

$$(\text{EFQ}^{ax})\ A, \neg A \Rightarrow \qquad (\text{EFQ}\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta}{A, \neg A, \Gamma \Rightarrow \Delta} \qquad (\neg\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}$$

$$(\neg\Rightarrow^{-1})\ \frac{\Gamma \Rightarrow \Delta, \neg A}{A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \text{EFQ})\ \frac{\Gamma \Rightarrow \Delta, A \quad \Pi \Rightarrow \Sigma, \neg A}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

[8] However, if we want to have this axiom restricted to propositional variables, then we have to consider two axioms: $p \Rightarrow p$ and $\neg p \Rightarrow \neg p$, for any propositional variable p.

Similarly, one can find several sequent analogous of the natural deductions rules ($\mathrm{EM}_\neg$) and ($\mathrm{EFQ}_\neg$): just add one more negation to A and $\neg A$ in the above presented analogous versions of (EM and (EFQ). Again, all versions are equivalent, but for constructive cut admissibility, the latter ones are needed (($\Rightarrow \mathrm{EM}_\neg$) and ($\Rightarrow \mathrm{EFQ}_\neg$)).

$$(\mathrm{EM}^{ax}_\neg) \Rightarrow \neg A, \neg\neg A \qquad (\Rightarrow \mathrm{EM}_\neg)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A, \neg\neg A} \qquad (\Rightarrow \neg\neg_{\mathbf{G}})\ \frac{\neg A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\neg A}$$

$$(\Rightarrow \neg_\neg^{-1})\ \frac{\neg\neg A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A} \qquad (\mathrm{EM}_\neg \Rightarrow)\ \frac{\neg A, \Gamma \Rightarrow \Delta \quad \neg\neg A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(\mathrm{EFQ}^{ax}_\neg)\ \neg A, \neg\neg A \Rightarrow \qquad (\mathrm{EFQ}_\neg \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta}{\neg A, \neg\neg A, \Gamma \Rightarrow \Delta} \qquad (\neg\neg_{\mathbf{G}} \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg A}{\neg\neg A, \Gamma \Rightarrow \Delta}$$

$$(\neg_\neg \Rightarrow^{-1})\ \frac{\Gamma \Rightarrow \Delta, \neg\neg A}{\neg A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \mathrm{EFQ}_\neg)\ \frac{\Gamma \Rightarrow \Delta, \neg A \quad \Pi \Rightarrow \Sigma, \neg\neg A}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

Notice that for the negation fragment of $\mathbf{DG_3}$ Avron [7] uses the rules ($\Rightarrow \neg$) and ($\neg\neg \Rightarrow_{\mathbf{G}}$).

The natural deduction rules for an n-ary connective $\circledcirc$ can be transformed into the following sequent rules (the case of $\mathbf{LP}$ and $\mathbf{DG_3}$):

$$(\neg\circledcirc \Rightarrow^{\mathbf{LPDG_3}}_{\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1})\ \frac{\{\neg A_i, \Gamma \Rightarrow \Delta\}^\dagger \quad \{\Gamma \Rightarrow \Delta, A_j\}^\ddagger \quad \{\Gamma \Rightarrow \Delta, \neg A_j\}^\ddagger \quad \{A_k, \Gamma \Rightarrow \Delta\}^*}{\neg\circledcirc(A_1, \ldots, A_n), \Gamma \Rightarrow \Delta}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

$$(\Rightarrow \circledcirc^{\mathbf{LPDG_3}}_{\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2})\ \frac{\{\neg A_i, \Gamma \Rightarrow \Delta\}^\dagger \quad \{\Gamma \Rightarrow \Delta, A_j\}^\ddagger \quad \{\Gamma \Rightarrow \Delta, \neg A_j\}^\ddagger \quad \{A_k, \Gamma \Rightarrow \Delta\}^*}{\Gamma \Rightarrow \Delta, \circledcirc(A_1, \ldots, A_n)}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

$$(\Rightarrow \neg\circledcirc^{\mathbf{LPDG_3}}_{\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2})\ \frac{\{\neg A_i, \Gamma \Rightarrow \Delta\}^\dagger \quad \{\Gamma \Rightarrow \Delta, A_j\}^\ddagger \quad \{\Gamma \Rightarrow \Delta, \neg A_j\}^\ddagger \quad \{A_k, \Gamma \Rightarrow \Delta\}^*}{\Gamma \Rightarrow \Delta, \neg\circledcirc(A_1, \ldots, A_n)}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

$$(\circledcirc \Rightarrow^{\mathbf{LPDG_3}}_{\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0})\ \frac{\{\neg A_i, \Gamma \Rightarrow \Delta\}^\dagger \quad \{\Gamma \Rightarrow \Delta, A_j\}^\ddagger \quad \{\Gamma \Rightarrow \Delta, \neg A_j\}^\ddagger \quad \{A_k, \Gamma \Rightarrow \Delta\}^*}{\circledcirc(A_1, \ldots, A_n), \Gamma \Rightarrow \Delta}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

The natural deduction rules for an n-ary connective $\circledcirc$ can be transformed into the following sequent rules (the case of $\mathbf{K_3}$ and $\mathbf{G_3}$):

$$(\Rightarrow \circledcirc^{\mathbf{K_3G_3}}_{\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1})\ \frac{\{\Gamma \Rightarrow \Delta, A_i\}^\dagger \quad \{A_j, \Gamma \Rightarrow \Delta\}^\ddagger \quad \{\neg A_j, \Gamma \Rightarrow \Delta\}^\ddagger \quad \{\Gamma \Rightarrow \Delta, \neg A_k\}^*}{\Gamma \Rightarrow \Delta, \circledcirc(A_1, \ldots, A_n)}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

$$(\circledcirc^{\mathbf{K_3G_3}}_{\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2} \Rightarrow)\ \frac{\{\Gamma \Rightarrow \Delta, A_i\}^\dagger \quad \{A_j, \Gamma \Rightarrow \Delta\}^\ddagger \quad \{\neg A_j, \Gamma \Rightarrow \Delta\}^\ddagger \quad \{\Gamma \Rightarrow \Delta, \neg A_k\}^*}{\circledcirc(A_1, \ldots, A_n), \Gamma \Rightarrow \Delta}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

$$(\neg\circledcirc^{\mathbf{K_3G_3}}_{\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2} \Rightarrow)\ \frac{\{\Gamma \Rightarrow \Delta, A_i\}^\dagger \quad \{A_j, \Gamma \Rightarrow \Delta\}^\ddagger \quad \{\neg A_j, \Gamma \Rightarrow \Delta\}^\ddagger \quad \{\Gamma \Rightarrow \Delta, \neg A_k\}^*}{\neg\circledcirc(A_1, \ldots, A_n), \Gamma \Rightarrow \Delta}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

$$(\Rightarrow \neg\circledcirc^{\mathbf{K_3G_3}}_{\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0})\ \frac{\{\Gamma \Rightarrow \Delta, A_i\}^\dagger \quad \{A_j, \Gamma \Rightarrow \Delta\}^\ddagger \quad \{\neg A_j, \Gamma \Rightarrow \Delta\}^\ddagger \quad \{\Gamma \Rightarrow \Delta, \neg A_k\}^*}{\Gamma \Rightarrow \Delta, \neg\circledcirc(A_1, \ldots, A_n)}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

The notion of the proof in these calculi is defined in a standard way.

Although a completeness proof is given in Section 4 (Theorems 142 and 143), together with modalities (we find such a way easier: to formulate the proof at once for Kripke semantics, rather than give it now and then adopt it for the use of possible worlds), a constructive cut elimination proof can be given already now.

Let us present a constructive cut admissibility proof for the logics in question.[9] We use the strategy by Metcalfe, Olivetti, and Gabbay [123] which we have already applied in Section 2.3.2. The notions of *principal formula*, *side formulas*, *parametric formulas*, the length $\mathfrak{l}(\mathfrak{D})$ of a derivation $\mathfrak{D}$, and the cut rank $\mathfrak{r}(\mathfrak{D})$ of a derivation $\mathfrak{D}$ are given in Section 2.3.2. The complexity $\mathfrak{c}(A)$ of a formula A is defined as a degree $d(A)$ of the formula A introduced in Definition 96.

Lemma 105 (Right reduction). *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be derivations such that*:

(1) *$\mathfrak{D}_1$ is a derivation of $\Gamma \Rightarrow \Delta, A$,*

(2) *$\mathfrak{D}_2$ is a derivation of $A^a, \Theta \Rightarrow \Lambda$,*

(3) *$\mathfrak{r}(\mathfrak{D}_1) \leq \mathfrak{c}(A)$ and $\mathfrak{r}(\mathfrak{D}_2) \leq \mathfrak{c}(A)$,*

(4) *A is the principal formula of a logical rule in $\mathfrak{D}_1$.*

Then we can construct a derivation $\mathfrak{D}_0$ of $\Gamma^a, \Theta \Rightarrow \Lambda, \Delta^a$ such that $\mathfrak{r}(\mathfrak{D}_0) \leq \mathfrak{c}(A)$.

Proof. By induction on $\mathfrak{l}(\mathfrak{D}_2)$.

Basic case. Case 1. Let $A^a, \Theta \Rightarrow \Lambda$ be an axiom $A \Rightarrow A$. What we need to obtain is $\Gamma \Rightarrow \Delta, A$ which we already have.

Inductive case. We have different cases depending on the last rule applied in $\mathfrak{D}_2$.

Consider the logic **LP**. *Case 1.* The rule of the last application in $\mathfrak{D}_2$ is $(\neg\neg \Rightarrow)$. *Subcase 1. A* is principal in $\mathfrak{D}_2$ and $A = \neg\neg B$, and the rules $(\Rightarrow \neg\neg)$ and $(\neg\neg \Rightarrow)$ are applied. The last inference of $\mathfrak{D}_2$ looks as follows.

$$\frac{B, \neg\neg B^a, \Theta \Rightarrow \Lambda}{\neg\neg B^{a+1}, \Theta \Rightarrow \Lambda}\ (\neg\neg \Rightarrow)$$

The last inference of $\mathfrak{D}_1$ looks as follows.

$$\frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, \neg\neg B}\ (\Rightarrow \neg\neg)$$

What we need to obtain is $\Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda$.

By the induction hypothesis, we have a derivation $\mathfrak{D}_3$ such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$: $B, \Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda$. Then we apply the cut to the formulas of a lower complexity:

$$\cfrac{\cfrac{\Gamma \Rightarrow \Delta, B \qquad B, \Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda}{\Gamma^{a+2}, \Theta \Rightarrow \Delta^{a+2}, \Lambda}}{\Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda}\ (\mathrm{C}\Rightarrow), (\Rightarrow\mathrm{C})$$

Subcase 1.2. A is not the principal formula in $\mathfrak{D}_2$. The last inference of $\mathfrak{D}_2$ looks as follows.

$$\frac{B, A^a, \Theta \Rightarrow \Lambda}{\neg\neg B, A^a, \Theta \Rightarrow \Lambda}\ (\neg\neg \Rightarrow)$$

[9]However, this proof does not lead us to the negation subformula property, because of the usage of the rules (EM$\Rightarrow$), ($\Rightarrow$EFQ), (EM$_\neg \Rightarrow$), and ($\Rightarrow$ EFQ$_\neg$). However, the negation subformula property is established by a Hintikka-style completeness argument in the next chapter.

The last sequent of $\mathfrak{D}_1$ is $\Gamma \Rightarrow \Delta, A$, what we need to obtain is $\neg\neg B, \Gamma^a, \Theta \Rightarrow \Delta^a, \Lambda$.

By the induction hypothesis, we have a derivation $\mathfrak{D}_3$ such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$: $B, \Gamma^a, \Theta \Rightarrow \Delta^a, \Lambda$. Then we apply the rule $(\neg\neg \Rightarrow)$:

$$\frac{B, \Gamma^a, \Theta \Rightarrow \Delta^a, \Lambda}{\neg\neg B, \Gamma^a, \Theta \Rightarrow \Delta^a, \Lambda}$$

Case 2. The rule of the last application in $\mathfrak{D}_2$ is $(\neg\circledcirc \Rightarrow^{\mathbf{LPDG_3}}_{\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1})$. *Subcase 2.1.* A is principal in $\mathfrak{D}_2$ and $A = \neg\circledcirc(B_1, \ldots, B_n) = \neg\circledcirc(\vec{B})$. The derivation $\mathfrak{D}_2$ is as follows.

$$\frac{\{A^a, \neg B_i, \Theta \Rightarrow \Lambda\}^{\dagger} \qquad \{A^a, \Theta \Rightarrow \Lambda, B_j\}^{\ddagger} \qquad \{A^a, \Theta \Rightarrow \Lambda, \neg B_j\}^{\ddagger} \qquad \{A^a, B_k, \Theta \Rightarrow \Lambda\}^{*}}{A^{a+1}, \Theta \Rightarrow \Lambda}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

The deduction $\mathfrak{D}_1$ is as follows (the rule $(\neg\circledcirc \Rightarrow^{\mathbf{LPDG_3}}_{\langle \mathfrak{t}',\mathfrak{h}',\mathfrak{f}'\rangle 1/2})$):

$$\frac{\{\neg B_{i'}, \Gamma \Rightarrow \Delta\}^{\sharp} \qquad \{\Gamma \Rightarrow \Delta, B_{j'}\}^{\S} \qquad \{\Gamma \Rightarrow \Delta, \neg B_{j'}\}^{\S} \qquad \{B_{k'}, \Gamma \Rightarrow \Delta\}^{\star}}{\Gamma \Rightarrow \Delta, A}$$

$^{\sharp}$ for each $i' \in \mathfrak{t}'$, § for each $j' \in \mathfrak{h}'$, * for each $k' \in \mathfrak{f}'$.

What we need to obtain is $\Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda$.

Let us recall that $\neg B_i, B_j, \neg B_j, B_k, \neg B_{i'}, B_{j'}, \neg B_{j'}, B_{k'}$ are either subformulas or negations of subformulas of $\neg\circledcirc(\vec{B}) = \neg\circledcirc(B_1, \ldots, B_n)$. We need to have more details about the other rules which are present in the system. Let us observe that this case can be reformulated in the following way ($\mathfrak{D}_1$ is displayed on the left, $\mathfrak{D}_2$ is displayed on the right):

$$\frac{X_1 \ldots X_n}{\Gamma \Rightarrow \Delta, A} \qquad \frac{Y_1 \ldots Y_n}{A^{a+1}, \Theta \Rightarrow \Lambda}$$

where $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are abbreviations for the following derivations, for any $t, u \in \{1, \ldots, n\}$:

$$X_t = \begin{cases} B_t, \Gamma \Rightarrow \Delta & \textit{iff} \quad t \in \mathfrak{f}'; \\ \left[\begin{array}{l} \Gamma \Rightarrow \Delta, B_t \\ \Gamma \Rightarrow \Delta, \neg B_t \end{array}\right. & \textit{iff} \quad t \in \mathfrak{h}'; \\ \neg B_t, \Gamma \Rightarrow \Delta & \textit{iff} \quad t \in \mathfrak{t}'; \end{cases} \qquad Y_u = \begin{cases} A^a, B_u, \Theta \Rightarrow \Lambda & \textit{iff} \quad u \in \mathfrak{f}; \\ \left[\begin{array}{l} A^a, \Theta \Rightarrow \Lambda, B_u \\ A^a, \Theta \Rightarrow \Lambda, \neg B_u \end{array}\right. & \textit{iff} \quad u \in \mathfrak{h}; \\ A^a, \neg B_u, \Theta \Rightarrow \Lambda & \textit{iff} \quad u \in \mathfrak{t}. \end{cases}$$

As follows from these equalities and soundness of our sequent calculi, there is $l \in \{1, \ldots, n\}$ such that $X_l \neq Y_l$.[10] The following combinations are possible (we present them in the form of ordered pairs $\langle X_l, Y_l \rangle$):

$$\mathcal{C}_1 = \left\langle\ B_l, \Gamma \Rightarrow \Delta\ ,\ \left[\begin{array}{l} A^a, \Theta \Rightarrow \Lambda, B_l \\ A^a, \Theta \Rightarrow \Lambda, \neg B_l \end{array}\right. \right\rangle \quad \mathcal{C}_2 = \left\langle\ B_l, \Gamma \Rightarrow \Delta\ ,\ A^a, \neg B_l, \Theta \Rightarrow \Lambda\ \right\rangle$$

$$\mathcal{C}_3 = \left\langle \left[\begin{array}{l} \Gamma \Rightarrow \Delta, B_l \\ \Gamma \Rightarrow \Delta, \neg B_l \end{array}\right. ,\ A^a, B_l, \Theta \Rightarrow \Lambda \right\rangle \quad \mathcal{C}_4 = \left\langle \left[\begin{array}{l} \Gamma \Rightarrow \Delta, B_l \\ \Gamma \Rightarrow \Delta, \neg B_l \end{array}\right. ,\ A^a, \neg B_l, \Theta \Rightarrow \Lambda \right\rangle$$

$$\mathcal{C}_5 = \left\langle\ \neg B_l, \Gamma \Rightarrow \Delta\ ,\ A^a, B_l, \Theta \Rightarrow \Lambda\ \right\rangle \quad \mathcal{C}_6 = \left\langle\ \neg B_l, \Gamma \Rightarrow \Delta\ ,\ \left[\begin{array}{l} A^a, \Theta \Rightarrow \Lambda, B_l \\ A^a, \Theta \Rightarrow \Lambda, \neg B_l \end{array}\right. \right\rangle$$

Let us consider the case $\mathcal{C}_1$. The derivation $\mathfrak{D}_1$ is as follows:

[10] Otherwise $\mathfrak{t} = \mathfrak{t}'$, $\mathfrak{h} = \mathfrak{h}'$, and $\mathfrak{f} = \mathfrak{f}'$ which implies that both $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle) = 1$ and $f_{\circledcirc}(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle) = 1/2$.

$$\frac{X_1 \ldots X_{l-1} \quad B_l, \Gamma \Rightarrow \Delta \quad X_{l+1} \ldots X_n}{\Gamma \Rightarrow \Delta, A}$$

The derivation $\mathfrak{D}_2$ is as follows:

$$\frac{Y_1 \ldots Y_{l-1} \quad A^a, \Theta \Rightarrow \Lambda, B_l \quad A^a, \Theta \Rightarrow \Lambda, \neg B_l \quad Y_{l+1} \ldots Y_n}{A^{a+1}, \Theta \Rightarrow \Lambda}$$

What we need is $\Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda$.

By the induction hypothesis, we have a derivation $\mathfrak{D}_3$ such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$: $\Theta, \Gamma^{a+1} \Rightarrow \Lambda, \Delta^{a+1}, B_l$. Then we apply the cut to the formulas of a lower complexity and structural rules to obtain the required result:

$$\cfrac{\cfrac{\Theta, \Gamma^{a+1} \Rightarrow \Lambda, \Delta^{a+1}, B_l \qquad B_l, \Gamma \Rightarrow \Delta}{\Gamma^{a+2}, \Theta \Rightarrow \Delta^{a+2}, \Lambda}}{\Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda}$$

The cases $\mathcal{C}_3$, $\mathcal{C}_4$, and $\mathcal{C}_6$ are considered similarly.

Let us consider the case $\mathcal{C}_2$. The last inference of $\mathfrak{D}_1$ is as follows:

$$\frac{X_1 \ldots X_{l-1} \quad B_l, \Gamma \Rightarrow \Delta \quad X_{l+1} \ldots X_n}{\Gamma \Rightarrow \Delta, A}$$

The last inference of $\mathfrak{D}_2$ is as follows:

$$\frac{Y_1 \ldots Y_{l-1} \quad A^a, \neg B_l, \Theta \Rightarrow \Lambda \quad Y_{l+1} \ldots Y_n}{A^{a+1}, \Theta \Rightarrow \Lambda}$$

What we need is $\Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda$.

By the induction hypothesis, we have a derivation $\mathfrak{D}_3$ such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$: $\neg B_l, \Gamma^{a+1}, \Theta \Rightarrow \Lambda, \Delta^{a+1}$. Then we apply the rule (EM$\Rightarrow$) and structural rules to obtain the required result:

$$\cfrac{\cfrac{B_l, \Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda \qquad \neg B_l, \Gamma^{a+1}, \Theta \Rightarrow \Lambda, \Delta^{a+1}}{\Gamma^{2(a+1)}, \Theta^2 \Rightarrow \Delta^{2(a+1)}, \Lambda^2}}{\Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda}$$

The case $\mathcal{C}_5$ is considered similarly.

Subcase 2.2. A is not principal in $\mathfrak{D}_2$ and $A \neq \neg\circledcirc(B_1, \ldots, B_n) = \neg\circledcirc(\vec{B})$. The derivation $\mathfrak{D}_2$ is as follows.

$$\frac{\{A^a, \neg B_i, \Theta \Rightarrow \Lambda\}^\dagger \qquad \{A^a, \Theta \Rightarrow \Lambda, B_j\}^\ddagger \qquad \{A^a, \Theta \Rightarrow \Lambda, \neg B_j\}^\ddagger \qquad \{A^a, B_k, \Theta \Rightarrow \Lambda\}^*}{A^{a+1}, \neg\circledcirc(\vec{B}), \Theta \Rightarrow \Lambda}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

The deduction $\mathfrak{D}_1$ is as follows: $\Gamma \Rightarrow \Delta, A$. What we need is as follows: $\neg\circledcirc(\vec{B}), \Gamma^a, \Theta \Rightarrow \Delta^a, \Lambda$. Using the inductive hypothesis, the rule $(\neg\circledcirc \Rightarrow^{\mathbf{LPDG_3}}_{\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1})$, we obtain

$$\frac{\{\neg B_i, \Gamma^a, \Theta \Rightarrow \Delta^a, \Lambda\}^\dagger \qquad \{\Gamma^a, \Theta \Rightarrow \Delta^a, \Lambda, B_j\}^\ddagger \qquad \{\Gamma^a, \Theta \Rightarrow \Delta^a, \Lambda, \neg B_j\}^\ddagger \qquad \{B_k, \Gamma^a, \Theta \Rightarrow \Delta^a, \Lambda\}^*}{\neg\circledcirc(\vec{B}), \Gamma^a, \Theta \Rightarrow \Delta^a, \Lambda}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

Case 3. The rule of the last application in $\mathfrak{D}_2$ is $(\circledcirc \Rightarrow^{\mathbf{LPDG_3}}_{\langle\mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0})$. Similarly to Case 2.

Case 4. The rule of the last application in $\mathfrak{D}_2$ is (EM$\Rightarrow$). The last inference of $\mathfrak{D}_2$ is as follows:

$$\frac{B, A^a, \Theta_1 \Rightarrow \Lambda_1 \qquad \neg B, A^a, \Theta_2 \Rightarrow \Lambda_2}{A^a, \Theta_1, \Theta_2 \Rightarrow \Lambda_1, \Lambda_2}$$

The last sequent of $\mathfrak{D}_1$ is $\Gamma \Rightarrow \Delta, A$. What we need is $\Gamma^a, \Theta_1, \Theta_2 \Rightarrow \Delta^a, \Lambda_1, \Lambda_2$. Using the induction hypothesis and the rule (EM$\Rightarrow$) we obtain the required result:

$$\frac{B, \Gamma^a, \Theta_1, \Theta_2 \Rightarrow \Delta^a, \Lambda_1, \Lambda_2 \qquad \neg B, \Gamma^a, \Theta_1, \Theta_2 \Rightarrow \Delta^a, \Lambda_1, \Lambda_2}{\Gamma^a, \Theta_1, \Theta_2 \Rightarrow \Delta^a, \Lambda_1, \Lambda_2}$$

The cases of logics $\mathbf{K_3}$, $\mathbf{DG_3}$, and $\mathbf{G_3}$ are treated similarly. □

Lemma 106 (Left reduction)**.** *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be derivations such that:*

(1) $\mathfrak{D}_1$ *is a derivation of* $\Gamma \Rightarrow \Delta, A^i$,

(2) $\mathfrak{D}_2$ *is a derivation of* $A, \Theta \Rightarrow \Lambda$,

(3) $\mathfrak{r}(\mathfrak{D}_1) \leq \mathfrak{c}(A)$ *and* $\mathfrak{r}(\mathfrak{D}_2) \leq \mathfrak{c}(A)$.

Then we can construct a derivation $\mathfrak{D}_0$ of $\Gamma, \Theta^i \Rightarrow \Lambda^i, \Delta$ such that $\mathfrak{r}(\mathfrak{D}_0) \leq \mathfrak{c}(A)$.

Proof. The proof is by induction on $\mathfrak{l}(\mathfrak{D}_1)$. Similarly to Lemma 105. □

Theorem 107 (Constructive elimination of cuts)**.** *Let $\mathbf{L} \in \{\mathbf{LP}_{\neg}^{(\circledcirc)m}, \mathbf{DG}_{\mathbf{3}\neg}^{(\circledcirc)m}, \mathbf{K}_{\mathbf{3}\neg}^{(\circledcirc)m}, \mathbf{G}_{\mathbf{3}\neg}^{(\circledcirc)m}\}$. If a derivation $\mathfrak{D}$ in $\mathbb{SC}_\mathbf{L}$ has an application of* (Cut)*, then it can be transformed into a cut-free derivation $\mathfrak{D}'$.*

Proof. Assume that a derivation $\mathfrak{D}$ in $\mathbb{SC}_\mathbf{L}$ has at least one application of (Cut), i.e., $\mathfrak{r}(\mathfrak{D}) > 0$. The proof proceeds by the double induction on $\langle \mathfrak{r}(\mathfrak{D}), \mathfrak{nr}(\mathfrak{D}) \rangle$, where $\mathfrak{nr}(\mathfrak{D})$ is the number of applications of (Cut) in $\mathfrak{D}$. Consider the uppermost application of (Cut) in $\mathfrak{D}$ with a cut rank $\mathfrak{r}(\mathfrak{D})$. We apply Lemma 106 to its premises and decrease either $\mathfrak{r}(\mathfrak{D})$ or $\mathfrak{nr}(\mathfrak{D})$. Then we can use the inductive hypothesis. □

3.6 Other three-valued logics and four-valued ones

The results obtained by correspondence analysis can be extended to other logics, including some different three-valued logics and four-valued ones. In this section, we would like to describe their semantics and natural deduction rules for their negations found by Omori and Wansing [141], and explain our choice of exactly those logics.

We applied correspondence analysis to the negation fragments of the chosen three-valued logics. So if we want to deal with other three-valued logics and also four-valued ones, we need mainly to think about negations and justify their choice. We think that Łukasiewicz-Kleene's, Heyting's, and Bochvar's negations are the closest three-valued negations to natural language. Also, they are the only three-valued negations, which coincide with the classical one being restricted to the set $\{1, 0\}$ of truth values. However, in the literature there are options, e.g. Post's famous negation [159], which is known as cyclic negation and which does not coincide with the classical one being restricted to the set $\{1, 0\}$: if $v(A) = 1$, then $v(\neg A) = 1/2$. One may find in the literature its converse [147] together with natural deduction systems for Post's logics and their so-called duals in the language with $\neg$, $\wedge$, and $\vee$ (and also a brief survey of other proof systems for Post's logics).

If we are talking about four-valued negations, then the most famous and convenient option is the de Morgan negation from Belnap-Dunn's logic **FDE**. Other options seem to be much less popular, although they exist (a bit later we will give truth tables for several such negations).

However, it is good to have some general approach to the choice of negation, providing some answer to the question of which connectives can be considered negations in a three- and four-valued setting. Such an approach we found in Omori and Wansing's paper [141] where the authors consider Dunn's semantics for **FDE**, **LP**, and $\mathbf{K_3}$.

> "According to Arnon Avron, the requirement that $\neg A$ is true iff A is false represents "the idea of falsehood within the language" [1, p. 160]. We shall keep this truth condition for negated formulas but abandon the classical understanding of falsity as untruth and instead

treat truth and falsity as two separate primitive semantical notions of equal importance. There is thus a clear sense in which the unary connectives in this paper written as $\neg$, sometimes with a subscript, can be seen as negations. However, there is now room for tweaking the falsity condition for negation. We will consider all combinations that are possible for **FDE**, **K3**, and **LP** in a classical metatheory. This gives us sixteen variants of **FDE**, four variants of **K3**, and four variants of **LP**." [141, p. 3] (the notation adjusted, in our bibliography Avron's paper mentioned here is [6]).

We would like to consider all these negations (some of them are Łukasiewicz-Kleene's, Heyting's, and Bochvar's ones) and show that the results obtained by correspondence analysis are applicable to the logics with these negations as well.

To begin with, let us introduce a four-valued semantics for **FDE** (in the language with $\neg$, $\wedge$, and $\vee$). The truth values are as follows: 1 (true), b (both true and false), n (neither true nor false), and 0 (false). The designated values are 1 and b. In what follows, we will use the following notation, for any formula A and any valuation v: $v(A) \in \mathsf{1}$ iff $v(A) \in \{1, b\}$, and $v(A) \in \mathsf{0}$ iff $v(A) \in \{0, b\}$. In the case of three-valued logics with $1/2$ being designated, just put $1/2 = b$, and $1/2 = n$ otherwise. In what follows, this notation will simplify some definitions, and as was shown by Omori and Sano [138], with its help, Belnap-style four-valued semantics can be transformed into Dunn-style two-valued one. Omori and Wansing [141] use both types of semantics, but for our purposes only the Belnap-style one is needed. The connectives of **FDE** are defined as follows[11]:

A	$\neg$
1	0
b	b
n	n
0	1

$\vee$	1	b	n	0
1	1	1	1	1
b	1	b	1	b
n	1	1	n	n
0	1	b	n	0

$\wedge$	1	b	n	0
1	1	b	n	0
b	b	b	0	0
n	n	0	n	0
0	0	0	0	0

For any finite (multi)sets of formulas Γ and Δ, any formula A:

- $\Gamma \models_{\mathbf{FDE}} A$ iff for any valuation v, if $v(B) \in \mathsf{1}$, for each $B \in \Gamma$, then $v(A) \in \mathsf{1}$.
- $\Gamma \models_{\mathbf{FDE}} \Delta$ iff for any valuation v, if $v(B) \in \mathsf{1}$, for each $B \in \Gamma$, then $v(C) \in \mathsf{1}$, for some $C \in \Delta$.

"By simple combinatorial considerations, the following sixteen operations exhaust the space of possible connectives that share the truth condition for negation" [141, p. 5]:

A	$\neg_1$	$\neg_2$	$\neg_3$	$\neg_4$	$\neg_5$	$\neg_6$	$\neg_7$	$\neg_8$
1	0	0	0	0	0	0	0	0
b	b	b	b	b	1	1	1	1
n	n	n	0	0	n	n	0	0
0	1	b	1	b	1	b	1	b

A	$\neg_9$	$\neg_{10}$	$\neg_{11}$	$\neg_{12}$	$\neg_{13}$	$\neg_{14}$	$\neg_{15}$	$\neg_{16}$
1	n	n	n	n	n	n	n	n
b	b	b	b	b	1	1	1	1
n	n	n	0	0	n	n	0	0
0	1	b	1	b	1	b	1	b

[11] **FDE** is built in the language $\mathscr{L}_{\neg\wedge\vee}$ with an alphabet consisting of the set of propositional variables $\mathcal{P}$, the connectives $\neg, \wedge, \vee$, left and right parenthesis. The set $\mathscr{F}_{\neg\wedge\vee}$ of all $\mathscr{L}_{\neg\wedge\vee}$-formulas is defined in a standard inductive way.

The negation denoted as $\neg_1$ is the negation of **FDE**, the negation $\neg_{16}$ was discussed by Omori and Wansing in [140].

In the case of $\mathbf{K_3}$-style logics, logics where $1/2$ is not designated, we put $n = 1/2$ and obtain the following four negations:

A	$\neg_1$	$\neg_2$	$\neg_3$	$\neg_4$
1	0	0	$1/2$	$1/2$
$1/2$	$1/2$	0	$1/2$	0
0	1	1	1	1

The negations $\neg_1$ and $\neg_2$, which have already been considered by us, are Łukasiewicz-Kleene's [115, 95] and Heyting's [74] negations, respectively. The negation $\neg_4$ is Post's above-mentioned negation [159].

In the case of **LP**-style logics, logics where $1/2$ is designated, we put $b = 1/2$ and obtain the following four negations:

A	$\neg_1$	$\neg_2$	$\neg_3$	$\neg_4$
1	0	0	0	0
$1/2$	$1/2$	$1/2$	1	1
0	1	$1/2$	1	$1/2$

The negations $\neg_1$ and $\neg_3$, which have been already considered by us, are Łukasiewicz-Kleene's [115, 95] and Bochvar's [17] negations, respectively. The negation $\neg_4$ is the above-mentioned converse of Post's negation [147].

We will write $\mathbf{FDE}^i$, $1 \leqslant i \leqslant 16$, $\mathbf{K_3^i}$, $1 \leqslant i \leqslant 4$, $\mathbf{LP}^i$, $1 \leqslant i \leqslant 4$, for the logics built in the language $\mathscr{L}_{\neg\wedge\vee}$, where $\neg$ is interpreted as $\neg^i$, and the entailment relation is interpreted as in **FDE**, $\mathbf{K_3}$, and **LP**, respectively.

Omori and Wansing [141] give natural deduction systems for these negations. The following system was used by them for **FDE** (the system for $\mathbf{K_3}$ has also the rule (EFQ) and the system for **LP** has also the rule (EM)):.[12]

$$(\vee I_1)\ \frac{A}{A \vee B} \qquad (\vee I_2)\ \frac{B}{A \vee B} \qquad (\vee E)\ \frac{A \vee B \quad \begin{array}{c}[A]\\ C\end{array} \quad \begin{array}{c}[B]\\ C\end{array}}{C}$$

$$(\wedge I)\ \frac{A \quad B}{A \wedge B} \qquad (\wedge E_1)\ \frac{A \wedge B}{A} \qquad (\wedge E_2)\ \frac{A \wedge B}{B}$$

$$(\neg\vee I)\ \frac{\neg A \quad \neg B}{\neg(A \vee B)} \qquad (\neg\vee E_1)\ \frac{\neg(A \vee B)}{\neg A} \qquad (\neg\vee E_2)\ \frac{\neg(A \vee B)}{\neg B}$$

$$(\neg\wedge I_1)\ \frac{\neg A}{\neg(A \wedge B)} \qquad (\neg\wedge I_2)\ \frac{\neg B}{\neg(A \wedge B)} \qquad (\neg \wedge E)\ \frac{\neg(A \wedge B) \quad \begin{array}{c}[\neg A]\\ C\end{array} \quad \begin{array}{c}[\neg B]\\ C\end{array}}{C}$$

$$(\neg\neg I)\ \frac{A}{\neg\neg A} \qquad (\neg\neg E)\ \frac{\neg\neg A}{A}$$

Omori and Wansing [141] propose the following rules for four-valued negations:

$$(\neg_1\neg_1 1)\ \frac{\neg_1\neg_1 A}{A} \qquad (\neg_1\neg_1 2)\ \frac{A}{\neg_1\neg_1 A}$$

[12]Normalisation was proved for these systems in [105].

$$(\neg_2\neg_2 1)\ \frac{\neg_2\neg_2 A}{A \vee \neg_2 A} \quad (\neg_2\neg_2 2)\ \frac{A}{\neg_2\neg_2 A} \quad (\neg_2\neg_2 3)\ \frac{\neg_2 A}{\neg_2\neg_2 A}$$

$$(\neg_3\neg_3 1)\ \frac{\neg_3\neg_3 A \quad \neg_3 A}{A} \quad (\neg_3\neg_3 2)\ \frac{A}{\neg_3\neg_3 A} \quad (\neg_3\neg_3 3)\ \frac{}{\neg_3 A \vee \neg_3\neg_3 A}$$

$$(\neg_4\neg_4)\ \frac{}{\neg_4\neg_4 A}$$

$$(\neg_5\neg_5 1)\ \frac{\neg_5 A \quad \neg_5\neg_5 A}{B} \quad (\neg_5\neg_5 2)\ \frac{\neg_5\neg_5 A}{A} \quad (\neg_5\neg_5 3)\ \frac{A}{\neg_5 A \vee \neg_5\neg_5 A}$$

$$(\neg_6\neg_6 1)\ \frac{A \quad \neg_6 A \quad \neg_6\neg_6 A}{B} \quad (\neg_6\neg_6 2)\ \frac{\neg_6\neg_6 A}{A \vee \neg_6 A} \quad (\neg_6\neg_6 3)\ \frac{A}{\neg_6 A \vee \neg_6\neg_6 A} \quad (\neg_6\neg_6 4)\ \frac{\neg_6 A}{A \vee \neg_6\neg_6 A}$$

$$(\neg_7\neg_7 1)\ \frac{\neg_7 A \quad \neg_7\neg_7 A}{B} \quad (\neg_7\neg_7 2)\ \frac{}{\neg_7 A \vee \neg_7\neg_7 A}$$

$$(\neg_8\neg_8 1)\ \frac{A \quad \neg_8 A \quad \neg_8\neg_8 A}{B} \quad (\neg_8\neg_8 2)\ \frac{}{A \vee \neg_8\neg_8 A} \quad (\neg_8\neg_8 3)\ \frac{}{\neg_8 A \vee \neg_8\neg_8 A}$$

$$(\neg_9\neg_9 1)\ \frac{\neg_9\neg_9 A}{A} \quad (\neg_9\neg_9 2)\ \frac{\neg_9\neg_9 A}{\neg_9 A} \quad (\neg_9\neg_9 3)\ \frac{A \quad \neg_9 A}{\neg_9\neg_9 A}$$

$$(\neg_{10}\neg_{10} 1)\ \frac{\neg_{10}\neg_{10} A}{\neg_{10} A} \quad (\neg_{10}\neg_{10} 2)\ \frac{\neg_{10} A}{\neg_{10}\neg_{10} A}$$

$$(\neg_{11}\neg_{11} 1)\ \frac{A \quad \neg_{11}\neg_{11} A}{\neg_{11} A} \quad (\neg_{11}\neg_{11} 2)\ \frac{\neg_{11} A \quad \neg_{11}\neg_{11} A}{A} \quad (\neg_{11}\neg_{11} 3)\ \frac{}{A \vee \neg_{11} A \vee \neg_{11}\neg_{11} A} \quad (\neg_{11}\neg_{11} 4)\ \frac{A \quad \neg_{11} A}{\neg_{11}\neg_{11} A}$$

$$(\neg_{12}\neg_{12} 1)\ \frac{A \quad \neg_{12}\neg_{12} A}{\neg_{12} A} \quad (\neg_{12}\neg_{12} 2)\ \frac{}{A \vee \neg_{12}\neg_{12} A} \quad (\neg_{12}\neg_{12} 3)\ \frac{\neg_{12} A}{\neg_{12}\neg_{12} A}$$

$$(\neg_{13}\neg_{13})\ \frac{\neg_{13}\neg_{13} A}{B}$$

$$(\neg_{14}\neg_{14} 1)\ \frac{A \quad \neg_{14}\neg_{14} A}{B} \quad (\neg_{14}\neg_{14} 2)\ \frac{\neg_{14}\neg_{14} A}{\neg_{14} A} \quad (\neg_{14}\neg_{14} 3)\ \frac{\neg_{14} A}{A \vee \neg_{14}\neg_{14} A}$$

$$(\neg_{15}\neg_{15} 1)\ \frac{A \quad \neg_{15}\neg_{15} A}{B} \quad (\neg_{15}\neg_{15} 2)\ \frac{\neg_{15} A \quad \neg_{15}\neg_{15} A}{B} \quad (\neg_{15}\neg_{15} 3)\ \frac{}{A \vee \neg_{15} A \vee \neg_{15}\neg_{15} A}$$

$$(\neg_{16}\neg_{16} 1)\ \frac{A \quad \neg_{16}\neg_{16} A}{B} \quad (\neg_{16}\neg_{16} 2)\ \frac{}{A \vee \neg_{16}\neg_{16} A}$$

Omori and Wansing [141] propose the following rules for three-valued negations, the case of $\mathbf{K}_3^i$, where $i \in \{1, 2, 3, 4\}$:

$$(\mathrm{EFQ})\ \frac{\begin{matrix}\mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_i A & A\end{matrix}}{B}$$

each logic has the rule (EFQ) as well as the corresponding rules from the following list:

$$(\neg_1\neg_1 1)\ \frac{\neg_1\neg_1 A}{A} \quad (\neg_1\neg_1 2)\ \frac{A}{\neg_1\neg_1 A} \quad (\neg_2\neg_2 1)\ \frac{}{\neg_2 A \vee \neg_2\neg_2 A}$$

$$(\neg_3\neg_3 1)\ \dfrac{\neg_3\neg_3 A}{B} \qquad (\neg_4\neg_4 1)\ \dfrac{A \quad \neg_4\neg_4 A}{B} \qquad (\neg_2\neg_2 2)\ \dfrac{}{A \vee \neg_4 A \vee \neg_4\neg_4 A}$$

Omori and Wansing [141] propose the following rules for three-valued negations, the case of $\mathbf{LP}^i$, where $i \in \{1, 2, 3, 4\}$:

$$(\mathrm{EM})^{a,b}\ \dfrac{\begin{array}{cc}[A]^a & [\neg_i A]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ B & B\end{array}}{B}$$

each logic has the rule (EM) as well as the corresponding rules from the following list:

$$(\neg_1\neg_1 1)\ \dfrac{\neg_1\neg_1 A}{A} \qquad (\neg_1\neg_1 2)\ \dfrac{A}{\neg_1\neg_1 A} \qquad (\neg_2\neg_2 1)\ \dfrac{}{\neg_2\neg_2 A}$$

$$(\neg_3\neg_3 1)\ \dfrac{\neg_3 A \quad \neg_3\neg_3 A}{B} \qquad (\neg_4\neg_4 1)\ \dfrac{A \quad \neg_4 A \quad \neg_4\neg_4 A}{B} \qquad (\neg_4\neg_4 2)\ \dfrac{}{A \vee \neg_4\neg_4 A}$$

These natural deduction systems for $\mathbf{FDE}^i$, $1 \leqslant i \leqslant 16$, $\mathbf{K}_{\mathbf{3}}^i$, $1 \leqslant i \leqslant 4$, and $\mathbf{LP}^i$, $1 \leqslant i \leqslant 4$ are shown to be sound and complete by Omori and Wansing [141].

In order to show that the above-described results for corresponding analysis for three-valued logics work for $\mathbf{K}_{\mathbf{3}}^i$, $1 \leqslant i \leqslant 4$, and $\mathbf{LP}^i$, $1 \leqslant i \leqslant 4$ as well as to describe correspondence analysis for $\mathbf{FDE}^i$, $1 \leqslant i \leqslant 16$, on the basis of Kooi and Tamminga's paper [97], and later prove the normalisation theorem for these logics, we need to avoid using $\vee$ and $\wedge$ for more generality and formulate the rules for negation as general introduction and elimination rules. Then we will be able to deal with the negation fragments of these logics and extend them by n-ary operators.

General introduction and elimination rules for four-valued negations:

$$(\neg_1\neg_1 E)^a\ \dfrac{\begin{array}{cc} & [A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_1\neg_1 A & B\end{array}}{B} \qquad (\neg_1\neg_1 I)^a\ \dfrac{\begin{array}{cc} & [\neg_1\neg_1 A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ A & B\end{array}}{B}$$

$$(\neg_2\neg_2 E)^{a,b}\ \dfrac{\begin{array}{ccc} & [A]^a & [\neg_2 A]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg_2\neg_2 A & B & B\end{array}}{B} \qquad (\neg_2\neg_2 I_1)^a\ \dfrac{\begin{array}{cc} & [\neg_2\neg_2 A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ A & B\end{array}}{B} \qquad (\neg_2\neg_2 I_2)^a\ \dfrac{\begin{array}{cc} & [\neg_2\neg_2 A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_2 A & B\end{array}}{B}$$

$$(\neg_3\neg_3 E)^a\ \dfrac{\begin{array}{ccc} & & [A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg_3\neg_3 A & \neg_3 A & B\end{array}}{B} \qquad (\neg_3\neg_3 I_1)^a\ \dfrac{\begin{array}{cc} & [\neg_3\neg_3 A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ A & B\end{array}}{B} \qquad (\neg_3\neg_3 I_2)^{a,b}\ \dfrac{\begin{array}{cc} [\neg_3 A]^a & [\neg_3\neg_3 A]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ B & B\end{array}}{B}$$

$$(\neg_4\neg_4 I)^a\ \dfrac{\begin{array}{c} [\neg_4\neg_4 A]^a \\ \mathfrak{D} \\ B\end{array}}{B}$$

$$(\neg_5\neg_5 E_1)\ \dfrac{\begin{array}{cc} \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_5\neg_5 A & \neg_5 A\end{array}}{B} \qquad (\neg_5\neg_5 E_2)^a\ \dfrac{\begin{array}{cc} & [\neg_5\neg_5 A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ A & B\end{array}}{B} \qquad (\neg_5\neg_5 I)^{a,b}\ \dfrac{\begin{array}{ccc} & [\neg_5 A]^a & [\neg_5\neg_5 A]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ A & B & B\end{array}}{B}$$

$$(\neg_6\neg_6 E_1)\ \dfrac{\begin{array}{ccc} \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg_6\neg_6 A & \neg_6 A & A \end{array}}{B} \qquad (\neg_6\neg_6 E_2)^{a,b}\ \dfrac{\begin{array}{ccc} & [A]^a & [\neg_6 A]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg_6\neg_6 A & B & B \end{array}}{B}$$

$$(\neg_6\neg_6 I_1)^{a,b}\ \dfrac{\begin{array}{ccc} & [\neg_6 A]^a & [\neg_6\neg_6 A]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ A & B & B \end{array}}{B} \qquad (\neg_6\neg_6 I_2)^{a,b}\ \dfrac{\begin{array}{ccc} & [A]^a & [\neg_6\neg_6 A]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg_6 A & B & B \end{array}}{B}$$

$$(\neg_7\neg_7 E_1)\ \dfrac{\begin{array}{cc} \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_7\neg_7 A & \neg_7 A \end{array}}{B} \qquad (\neg_7\neg_7 I)^{a,b}\ \dfrac{\begin{array}{cc} [\neg_7 A]^a & [\neg_7\neg_7 A]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ B & B \end{array}}{B}$$

$$(\neg_8\neg_8 E)\ \dfrac{\begin{array}{ccc} \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg_8\neg_8 A & \neg_8 A & A \end{array}}{B} \qquad (\neg_8\neg_8 I_1)^{a,b}\ \dfrac{\begin{array}{cc} [A]^a & [\neg_8\neg_8 A]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ B & B \end{array}}{B} \qquad (\neg_8\neg_8 I_2)^{a,b}\ \dfrac{\begin{array}{cc} [\neg_8 A]^a & [\neg_8\neg_8 A]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ B & B \end{array}}{B}$$

$$(\neg_9\neg_9 E_1)^{a}\ \dfrac{\begin{array}{cc} & [A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_9\neg_9 A & B \end{array}}{B} \qquad (\neg_9\neg_9 E_2)^{a}\ \dfrac{\begin{array}{cc} & [\neg_9 A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_9\neg_9 A & B \end{array}}{B} \qquad (\neg_9\neg_9 I)^{a}\ \dfrac{\begin{array}{ccc} & & [\neg_9\neg_9 A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ A & \neg_9 A & B \end{array}}{B}$$

$$(\neg_{10}\neg_{10} E)^{a}\ \dfrac{\begin{array}{cc} & [\neg_{10} A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_{10}\neg_{10} A & B \end{array}}{B} \qquad (\neg_{10}\neg_{10} I)^{a}\ \dfrac{\begin{array}{cc} & [\neg_{10}\neg_{10} A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_{10} A & B \end{array}}{B}$$

$$(\neg_{11}\neg_{11} E_1)^{a}\ \dfrac{\begin{array}{ccc} & & [\neg_{11} A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg_{11}\neg_{11} A & A & B \end{array}}{B} \qquad (\neg_{11}\neg_{11} E_2)^{a}\ \dfrac{\begin{array}{ccc} & & [A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg_{11}\neg_{11} A & \neg_{11} A & B \end{array}}{B}$$

$$(\neg_{11}\neg_{11} I_1)^{a,b,c}\ \dfrac{\begin{array}{ccc} [A]^a & [\neg_{11} A]^b & [\neg_{11}\neg_{11} A]^c \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ B & B & B \end{array}}{B} \qquad (\neg_{11}\neg_{11} I_2)^{a,b,c}\ \dfrac{\begin{array}{ccc} & & [\neg_{11}\neg_{11} A]^c \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ A & \neg_{11} A & B \end{array}}{B}$$

$$(\neg_{12}\neg_{12} I_1)^{a}\ \dfrac{\begin{array}{ccc} & & [\neg_{12} A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg_{12}\neg_{12} A & A & B \end{array}}{B} \qquad (\neg_{12}\neg_{12} I_2)^{a,b}\ \dfrac{\begin{array}{cc} [A]^a & [\neg_{12}\neg_{12} A]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ B & B \end{array}}{B} \qquad (\neg_{12}\neg_{12} I_1)^{a}\ \dfrac{\begin{array}{cc} & [\neg_{12}\neg_{12} A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_{12} A & B \end{array}}{B}$$

$$(\neg_{13}\neg_{13} E)\ \dfrac{\neg_{13}\neg_{13} A}{B}$$

$$(\neg_{14}\neg_{14} E_1)\ \dfrac{\begin{array}{cc} \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_{14}\neg_{14} A & A \end{array}}{B} \qquad (\neg_{14}\neg_{14} E_2)^{a}\ \dfrac{\begin{array}{cc} & [\neg_{14} A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_{14}\neg_{14} A & B \end{array}}{B} \qquad (\neg_{14}\neg_{14} I)^{a}\ \dfrac{\begin{array}{ccc} & [A]^a & [\neg_{14}\neg_{14} A]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg_{14} A & B & B \end{array}}{B}$$

$$(\neg_{15}\neg_{15}E_1)\ \dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2\\ \neg_{15}\neg_{15}A & A\end{array}}{B} \qquad (\neg_{15}\neg_{15}E_2)\ \dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2\\ \neg_{15}\neg_{15}A & \neg_{15}A\end{array}}{B} \qquad (\neg_{15}\neg_{15}I)^{a,b,c}\ \dfrac{\begin{array}{ccc}[A]^a & [\neg_{15}A]^b & [\neg_{15}\neg_{15}A]^c\\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3\\ B & B & B\end{array}}{B}$$

$$(\neg_{16}\neg_{16}E)\ \dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2\\ \neg_{16}\neg_{16}A & A\end{array}}{B} \qquad (\neg_{16}\neg_{16}I)^{a,b}\ \dfrac{\begin{array}{cc}[A]^a & [\neg_{16}\neg_{16}A]^b\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ B & B\end{array}}{B}$$

General introduction and elimination rules for three-valued negations, the case of $\mathbf{K}^i_{\mathbf{3}}$, where $i \in \{1, 2, 3, 4\}$:

$$(\mathrm{EFQ})\ \dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2\\ \neg_i A & A\end{array}}{B}$$

$$(\neg_1\neg_1E)^a\ \dfrac{\begin{array}{cc} & [A]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ \neg_1\neg_1A & B\end{array}}{B} \qquad (\neg_1\neg_1I)^a\ \dfrac{\begin{array}{cc} & [\neg_1\neg_1A]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ A & B\end{array}}{B} \qquad (\neg_2\neg_2I)^{a,b}\ \dfrac{\begin{array}{cc}[\neg_2A]^a & [\neg_2\neg_2A]^b\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ B & B\end{array}}{B}$$

$$(\neg_3\neg_3E)^a\ \dfrac{\begin{array}{c}\mathfrak{D}_1\\ \neg_3\neg_3A\end{array}}{B} \qquad (\neg_4\neg_4E)\ \dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2\\ \neg_4\neg_4A & A\end{array}}{B} \qquad (\neg_4\neg_4I)^{a,b,c}\ \dfrac{\begin{array}{ccc}[A]^a & [\neg_4A]^a & [\neg_4\neg_4A]^c\\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3\\ B & B & B\end{array}}{B}$$

General introduction and elimination rules for three-valued negations, the case of $\mathbf{LP}^i$, where $i \in \{1, 2, 3, 4\}$:

$$(\mathrm{EM})^{a,b}\ \dfrac{\begin{array}{cc}[A]^a & [\neg_q A]^b\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ B & B\end{array}}{B}$$

$$(\neg_1\neg_1E)^a\ \dfrac{\begin{array}{cc} & [A]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ \neg_1\neg_1A & B\end{array}}{B} \qquad (\neg_1\neg_1I)^a\ \dfrac{\begin{array}{cc} & [\neg_1\neg_1A]^a\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ A & B\end{array}}{B} \qquad (\neg_2\neg_2I)^a\ \dfrac{\begin{array}{c}[\neg_2\neg_2A]^a\\ \mathfrak{D}_1\\ B\end{array}}{B}$$

$$(\neg_3\neg_3E)\ \dfrac{\begin{array}{cc}\mathfrak{D}_1 & \mathfrak{D}_2\\ \neg_3\neg_3A & \neg_3A\end{array}}{B} \qquad (\neg_4\neg_4E)\ \dfrac{\begin{array}{ccc}\mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3\\ \neg_4\neg_4A & \neg_4A & A\end{array}}{B} \qquad (\neg_4\neg_4I)^{a,b}\ \dfrac{\begin{array}{cc}[A]^a & [\neg_4\neg_4A]^b\\ \mathfrak{D}_1 & \mathfrak{D}_2\\ B & B\end{array}}{B}$$

3.6.1 Natural deduction for other three-valued logics

Let us begin with the adaptation of correspondence analysis for additional three-valued logics taken from Omori and Wansing's research. The adaptation is based on the observation that Theorems 76 and 80 hold for these additional logics as well; their proofs even do not need any changes. All crucial properties for their proof properties of the negation in question hold for all three-valued negations described in [141].

Let us write $\mathbf{K}^i_{\mathbf{3}\neg}$, $1 \leqslant i \leqslant 4$, $\mathbf{LP}^i_{\neg}$, $1 \leqslant i \leqslant 4$, and $\mathbf{FDE}^i_{\neg}$, $1 \leqslant i \leqslant 16$, for the negation fragments of the logics $\mathbf{K}^i_{\mathbf{3}}$, $1 \leqslant i \leqslant 4$, $\mathbf{LP}^i$, $1 \leqslant i \leqslant 4$, and $\mathbf{FDE}^i$, $1 \leqslant i \leqslant 16$. Let us write $\mathbf{K}^{i(\circledcirc)_m}_{\mathbf{3}\neg}$, $1 \leqslant i \leqslant 4$, $\mathbf{LP}^{i(\circledcirc)_m}_{\neg}$, $1 \leqslant i \leqslant 4$, and $\mathbf{FDE}^{i(\circledcirc)_m}_{\neg}$, $1 \leqslant i \leqslant 16$, for the extensions of the above-mentioned negation fragments by n-ary connectives $\circledcirc_1, \ldots, \circledcirc_m$, where $m > 0$.

Theorem 108. *Let* $\mathbf{L}$ *be* $\mathbf{LP}_{\neg}^{i(\circledcirc)_m}$, $1 \leqslant i \leqslant 4$. *Then*:

(1) $f_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle\big) = 0$ *iff* $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 0\big)$ *is sound in* $\mathbf{L}$.
(2) $f_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle\big) = 1/2$ *iff both* $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1/2\big)$ *and* $R^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1/2\big)$ *are sound in* $\mathbf{L}$.
(3) $f_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle\big) = 1$ *iff* $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1\big)$ *is sound in* $\mathbf{L}$.

Proof. The proof coincides with the proof of Theorem 76. □

Theorem 109. *Let* $\mathbf{L}$ *be* $\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}$, $1 \leqslant i \leqslant 4$. *Then*:

(1) $f_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle\big) = 0$ *iff* $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 0\big)$ *is sound in* $\mathbf{L}$.
(2) $f_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle\big) = 1/2$ *iff both* $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1/2\big)$ *and* $\widetilde{R}^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1/2\big)$ *are sound in* $\mathbf{L}$.
(3) $f_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle\big) = 1$ *iff* $\widetilde{R}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{h}, \mathfrak{f}\rangle 1\big)$ *is sound in* $\mathbf{L}$.

Proof. The proof coincides with the proof of Theorem 80. □

Theorem 110 (Soundness)**.** *Let* $\mathbf{L} \in \{\mathbf{K_{3}}_{\neg}{}^{i(\circledcirc)_m}, \mathbf{LP}_{\neg}{}^{i(\circledcirc)_m}\}$. *Let* $\Gamma \subseteq \mathscr{F}^{\neg}_{(\circledcirc)_m}$ *and* $A \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$. *Then* $\Gamma \vdash_{\mathbf{L}} A$ *implies* $\Gamma \models_{\mathbf{L}} A$.

Proof. By induction on the length of the derivation. Use Theorems 108 and 109 and the fact that the general introduction/elimination rules for negations are sound (which can be established by an easy check). □

As for completeness, it is fully analogous to the cases of the logics $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$, $\mathbf{DG}_{3\neg}^{(\circledcirc)_m}$, $\mathbf{G}_{3\neg}^{(\circledcirc)_m}$, and $\mathbf{K}_{3\neg}^{(\circledcirc)_m}$. Let us just give one remark: the notions of theories should be adapted for new logics (except the logic $\mathbf{K}_{\mathbf{3}\neg}^{3(\circledcirc)_m}$, it needs a $\mathbf{K}_{\mathbf{3}\neg}^{(\circledcirc)_m}$-theory from Definition 85).

Definition 111. An $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$-theory Γ (see Definition 84) is said to be an $\mathbf{LP}_{\neg}^{2(\circledcirc)_m}$*-theory* iff it satisfies the following condition, for each $A \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$:

- $\neg\neg A \in \Gamma$.

Definition 112. An $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$-theory Γ (see Definition 84) is said to be an $\mathbf{LP}_{\neg}^{4(\circledcirc)_m}$*-theory* iff it satisfies the following condition, for each $A \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$:

- $A \in \Gamma$ or $\neg\neg A \in \Gamma$.

Definition 113. An $\mathbf{K}_{\mathbf{3}\neg}^{(\circledcirc)_m}$-theory Γ (see Definition 85) is said to be an $\mathbf{K}_{\mathbf{3}\neg}^{4(\circledcirc)_m}$*-theory* iff it satisfies the following condition, for each $A \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$:

- $A \in \Gamma$ or $\neg A \in \Gamma$ or $\neg\neg A \in \Gamma$.

Lemma 114. *Every* $\mathbf{K}_{\mathbf{3}\neg}^{4(\circledcirc)_m}$*-theory* Γ *satisfies the following condition, for each* $A \in \mathscr{F}_{(\circledcirc)_m}$:

- $A \notin \Gamma$ or $\neg\neg A \notin \Gamma$.

Proof. Similarly to Lemma 86, assume that the condition does not hold and using the rule $(\neg_4\neg_4 E)$ obtain a contradiction. □

Lemma 115. *Every* $\mathbf{K}_{\mathbf{3}\neg}^{3(\circledcirc)_m}$*-theory* Γ *satisfies the following condition, for each* $\alpha \in \mathscr{F}_{(\circledcirc)_m}$:

- $\neg\neg A \notin \Gamma$.

Proof. Similarly to Lemma 86, assume that the condition does not hold and using the rule $(\neg_3\neg_3 E)$ obtain a contradiction. □

Theorem 116 (Completeness)**.** *Let* $\mathbf{L} \in \{\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}, \mathbf{LP}_{\neg}^{i(\circledcirc)_m}\}$*. Let* $\Gamma \subseteq \mathscr{F}^{\neg}_{(\circledcirc)_m}$ *and* $A \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$*. Then* $\Gamma \models_{\mathbf{L}} A$ *implies* $\Gamma \vdash_{\mathbf{L}} A$*.*

Proof. Similarly to Theorem 6, using the above described definitions and lemmas. □

As for the normalisation proof, it is fully analogous to the cases of the logics $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$, $\mathbf{DG}_{\mathbf{3}\neg}^{(\circledcirc)_m}$, $\mathbf{G}_{\mathbf{3}\neg}^{(\circledcirc)_m}$, and $\mathbf{K}_{\mathbf{3}\neg}^{(\circledcirc)_m}$. Fortunately, the rules for n-ary connectives are the same, the only minor differences are connected with the rules for negations.

Theorem 117. *Let* $\mathbf{L} \in \{\mathbf{K}_{\mathbf{3}\neg}^{i}, \mathbf{LP}_{\neg}^{i}, \mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}, \mathbf{LP}_{\neg}^{i(\circledcirc)_m}\}$*,* $1 \leqslant i \leqslant 4$*. Any deduction in* $\mathbf{L}$ *can be converted into a deduction in normal form.*

Proof. Similarly to Theorem 100. □

Theorem 118. *Let* $\mathbf{L} \in \{\mathbf{K}_{\mathbf{3}\neg}^{i}, \mathbf{LP}_{\neg}^{i}, \mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}, \mathbf{LP}_{\neg}^{i(\circledcirc)_m}\}$*,* $1 \leqslant i \leqslant 4$*. Deductions in normal forms in* $\mathbf{L}$ *have the negation subformula property.*

Proof. Similarly to Theorem 104. □

3.6.2 Sequent calculi for other three-valued logics

Below is the list of the rules and axioms for negations for the logics $\mathbf{LP}_{\neg}^{i(\circledcirc)_m}$, $1 \leqslant i \leqslant 4$ (all the calculi have the axiom $A \Rightarrow A$[13], and the rule (EM⇒):

$$(\neg_1\neg_1 \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta}{\neg_1\neg_1 A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_1\neg_1)\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg_1\neg_1 A} \qquad (\Rightarrow \neg_2\neg_2)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_2\neg_2 A}$$

$$(\neg_3\neg_3 \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_3 A}{\neg_3\neg_3 A, \Gamma \Rightarrow \Delta} \qquad (\neg_4\neg_4 \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, \neg_4 A}{\neg_4\neg_4 A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_4\neg_4)\ \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_4\neg_4 A}$$

Below is the list of the rules and axioms for negations for the logics $\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}$, $1 \leqslant i \leqslant 4$ (all the calculi have the axioms $A \Rightarrow A$[14], and the rule (⇒EFQ):

$$(\neg_1\neg_1 \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta}{\neg_1\neg_1 A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_1\neg_1)\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \neg_1\neg_1 A} \qquad (\Rightarrow \neg_2\neg_2)\ \frac{\neg_2 A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_2\neg_2 A}$$

$$(\neg_3\neg_3)\ \frac{\Gamma \Rightarrow \Delta}{\neg_3\neg_3 A, \Gamma \Rightarrow \Delta} \qquad (\neg_4\neg_4 \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A}{\neg_4\neg_4 A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_4\neg_4)\ \frac{A, \Gamma \Rightarrow \Delta \quad \neg_4 A, \Pi \Rightarrow \Sigma}{\neg_4\neg_4 A, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

The rules for $\circledcirc$ were listed in Section 3.5.

Theorem 119. *Let* $\mathbf{L} \in \{\mathbf{LP}_{\neg}^{i(\circledcirc)_m}, \mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}\}$*,* $1 \leqslant i \leqslant 4$*. Let* $\Gamma \subseteq \mathscr{F}^{\neg}_{(\circledcirc)_m}$ *and* $A \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$*. Then:*

(1) $\Gamma \vdash_{\mathbb{ND}_{\mathbf{L}}} A$ *iff* $\vdash_{\mathbb{SC}_{\mathbf{L}}} \Gamma \Rightarrow A$*.*

(2) $\vdash_{\mathbb{SC}_{\mathbf{L}}} \Gamma \Rightarrow A$ *iff* $\Gamma \models_{\mathbf{L}} A$*.*

Proof. (1) By induction on the height of the derivation. (2) Follows from (1) and Theorem 116 (soundness and completeness of the natural deduction systems). □

In Chapter 4, we provide a Hintikka-style completeness proof for modal many-valued logics, including those that are modal extensions of the many-valued logics mentioned in Theorem 119, as a consequence, we obtain cut admissibility for the logics in question.

[13] It can be replaced by $p \Rightarrow p$ and $\neg_i p \Rightarrow \neg_i p$, for any propositional variable p.

[14] As in the previous case, it can be replaced by $p \Rightarrow p$ and $\neg_i p \Rightarrow \neg_i p$, for any propositional variable p.

3.6.3 Sequent calculi for four-valued logics

As for the four-valued logics $\mathbf{FDE}_{\neg}^{i(\circledcirc)_m}$, $1 \leqslant i \leqslant 16$, we will apply correspondence analysis to them in a different order: first sequent calculi and then natural deduction. The reason for such decision is as follows: we will use Kooi and Tamminga's [97] sequent calculus for $\mathbf{FDE}_{\neg}^{1(\circledcirc)_m}$, that is $\mathbf{FDE}_{\neg}^{(\circledcirc)_m}$ (recall that $\neg_1$ is the negation of **FDE**). We will show that their results can be easily generalised to the other fifteen four-valued logics. Then we convert these results into the natural deduction framework and prove the normalisation theorem.

Let us introduce the following lemmas from [97] which follow from the definition of the entailment relation in $\mathbf{FDE}_{\neg}^{(\circledcirc)_m}$ and the definition of its negation.

Lemma 120. *[97, Lemma 1] Let Γ and Δ be finite multisets of formulas, and A be a formula. Then:*

$$\Gamma \models \Delta, A \quad \textit{iff} \quad \textit{for every } \vartheta \textit{ such that } 1 \in \vartheta(\Gamma) \textit{ and } 1 \notin \vartheta(\Delta) \textit{ it holds that } 1 \in \vartheta(A),$$

$$\Gamma, A \models \Delta \quad \textit{iff} \quad \textit{for every } \vartheta \textit{ such that } 1 \in \vartheta(\Gamma) \textit{ and } 1 \notin \vartheta(\Delta) \textit{ it holds that } 1 \notin \vartheta(A)$$

Lemma 121. *[97, Lemma 2] Let A be a formula and ϑ be a valuation. Then:*

$$\vartheta(A) = n \quad \textit{iff} \quad 1 \notin \vartheta(A) \textit{ and } 1 \notin \vartheta(\neg A),$$

$$\vartheta(A) = 0 \quad \textit{iff} \quad 1 \notin \vartheta(A) \textit{ and } 1 \in \vartheta(\neg A),$$

$$\vartheta(A) = 1 \quad \textit{iff} \quad 1 \in \vartheta(A) \textit{ and } 1 \notin \vartheta(\neg A),$$

$$\vartheta(A) = b \quad \textit{iff} \quad 1 \in \vartheta(A) \textit{ and } 1 \in \vartheta(\neg A).$$

Let us make the following important observation which is a key to the generalisation of Kooi and Tamminga's results.

Lemma 122. *The statements of Lemmas 120 and 121 hold not only for $\mathbf{FDE}_{\neg}^{(\circledcirc)_m}$, but for $\mathbf{FDE}_{\neg}^{i(\circledcirc)_m}$ as well, for any $1 \leqslant i \leqslant 16$.*

Proof. Follows from the definition of the entailment relation in $\mathbf{FDE}_{\neg}^{i(\circledcirc)_m}$, $1 \leqslant i \leqslant 16$, and the definition of its negation $\neg_i$. □

Definition 123. [97, Definition 1] Let Γ and Δ be finite multisets of formulas, A be a formula, $z \in \{1, b, n, 0\}$. Then $A|_z^+$ and $A|_z^-$ are defined as follows:

$$A|_z^+(\Gamma \Rightarrow \Delta) = \begin{cases} \Gamma, A \Rightarrow \Delta & \text{if } z \in \{n, 0\}, \\ \Gamma \Rightarrow \Delta, A & \text{otherwise} \end{cases} \qquad A|_z^-(\Gamma \Rightarrow \Delta) = \begin{cases} \Gamma, \neg A \Rightarrow \Delta & \text{if } z \in \{n, 1\}, \\ \Gamma \Rightarrow \Delta, \neg A & \text{otherwise} \end{cases}$$

Lemma 124. *[97, Lemma 3] Let Γ and Δ be finite multisets of formulas, A be a formula, $z \in \{1, b, n, 0\}$. Then $A|_z^+(\Gamma \Rightarrow \Delta)$ and $A|_z^-(\Gamma \Rightarrow \Delta)$ are valid if and only if for every valuation ϑ such that $1 \in \vartheta(\Gamma)$ and $1 \notin \vartheta(\Delta)$, it holds that $\vartheta(A) = z$.*

As Kooi and Tamminga [97] note, this Lemma follows from Lemmas 120 and 121. But it also can be generalised for the other fifteen logics.

Lemma 125. *The statement of Lemma 124 holds not only for $\mathbf{FDE}_{\neg}^{(\circledcirc)_m}$, but for $\mathbf{FDE}_{\neg}^{i(\circledcirc)_m}$ as well, for any $1 \leqslant i \leqslant 16$.*

Proof. Follows from Lemma 122. □

Definition 126. [97, Definition 2] Let $\circledcirc$ be an n-ary operator and let $f_{\circledcirc}(x_1, \ldots, x_n) = y$ be a truth table entry. Then $R^{\circledcirc+}_{x_1,\ldots,x_n,y}$ and $R^{\circledcirc-}_{x_1,\ldots,x_n,y}$ are the following sequent rules:

$$R^{\circledcirc+}_{x_1,\ldots,x_n,y}\ \frac{A_1|^+_{x_1}(\Gamma\Rightarrow\Delta)\quad A_1|^-_{x_1}(\Gamma\Rightarrow\Delta)\quad\ldots\quad A_n|^+_{x_n}(\Gamma\Rightarrow\Delta)\quad A_n|^-_{x_n}(\Gamma\Rightarrow\Delta)}{\circledcirc(A_1,\ldots,A_n)|^+_y(\Gamma\Rightarrow\Delta)}$$

$$R^{\circledcirc-}_{x_1,\ldots,x_n,y}\ \frac{A_1|^+_{x_1}(\Gamma\Rightarrow\Delta)\quad A_1|^-_{x_1}(\Gamma\Rightarrow\Delta)\quad\ldots\quad A_n|^+_{x_n}(\Gamma\Rightarrow\Delta)\quad A_n|^-_{x_n}(\Gamma\Rightarrow\Delta)}{\circledcirc(A_1,\ldots,A_n)|^-_y(\Gamma\Rightarrow\Delta)}$$

Theorem 127. *[97, Theorem 1]*
$f_\circledcirc(x_1,\ldots,x_n)=y$ *iff both* $R^{\circledcirc+}_{x_1,\ldots,x_n,y}$ *and* $R^{\circledcirc-}_{x_1,\ldots,x_n,y}$ *are sound in* $\mathbf{FDE}^{(\circledcirc)_m}_\neg$.

Theorem 128. *Let* $1\leqslant i\leqslant 16$.
$f_\circledcirc(x_1,\ldots,x_n)=y$ *iff both* $R^{\circledcirc+}_{x_1,\ldots,x_n,y}$ *and* $R^{\circledcirc-}_{x_1,\ldots,x_n,y}$ *are sound in* $\mathbf{FDE}^{1(\circledcirc)_m}_\neg$.

Proof. Similarly to [97, Theorem 1] and Theorems 76 and 80, using Lemma 125. □

Kooi and Tamminga [97] consider the following sequent calculus for the negation fragment of **FDE** (sequents are built from multisets of formulas). It has the following axiom:

$$A,\Gamma\Rightarrow\Delta,A$$

The structural rules are as follows: contraction and weakening[15]. The logical rules are as follows: $(\neg\neg\Rightarrow)$ and $(\Rightarrow\neg\neg)$. This system has no cut rule.

The sequent calculus for $\mathbf{FDE}^{(\circledcirc)_m}_\neg$ is obtained from the sequent calculus for the negation fragment of **FDE** by adding the rules $R^{\circledcirc_t+}_{x_1,\ldots,x_n,y}$ and $R^{\circledcirc_t-}_{x_1,\ldots,x_n,y}$ for each $\circledcirc_t$, where $1\leqslant t\leqslant m$ and $m>0$.

Sequent rules for the other negations can be obtained from Omori and Wansing's [141] natural deduction rules for negations (or their general introduction and elimination versions). These sequents rules form sequent calculi for $\mathbf{FDE}^i_\neg$, $1\leqslant i\leqslant 16$, their extensions by the rules $R^{\circledcirc_t+}_{x_1,\ldots,x_n,y}$ and $R^{\circledcirc_t-}_{x_1,\ldots,x_n,y}$ for each $\circledcirc_t$, where $1\leqslant t\leqslant m$ and $m>0$, form sequent calculi for $\mathbf{FDE}^{i(\circledcirc)_m}_\neg$, for any $1\leqslant i\leqslant 16$.

Below is a list of the rules and axioms for the negations:

$$(\neg_1\neg_1\Rightarrow)\ \frac{A,\Gamma\Rightarrow\Delta}{\neg_1\neg_1A,\Gamma\Rightarrow\Delta}\qquad(\Rightarrow\neg_1\neg_1)\ \frac{\Gamma\Rightarrow\Delta,A}{\Gamma\Rightarrow\Delta,\neg_1\neg_1A}$$

$$(\neg_2\neg_2\Rightarrow)\ \frac{A,\Gamma\Rightarrow\Delta\quad\neg_2A,\Pi\Rightarrow\Sigma}{\neg_2\neg_2A,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\qquad(\Rightarrow\neg_2\neg_2)\ \frac{\Gamma\Rightarrow\Delta,A}{\Gamma\Rightarrow\Delta,\neg_2\neg_2A}\qquad(\Rightarrow\neg_2\neg_2^*)\ \frac{\Gamma\Rightarrow\Delta,\neg_2A}{\Gamma\Rightarrow\Delta,\neg_2\neg_2A}$$

$$(\neg_3\neg_3\Rightarrow)\ \frac{A,\Gamma\Rightarrow\Delta\quad\Pi\Rightarrow\Sigma,\neg_3A}{\neg_3\neg_3A,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\qquad(\Rightarrow\neg_3\neg_3)\ \frac{\Gamma\Rightarrow\Delta,A}{\Gamma\Rightarrow\Delta,\neg_3\neg_3A}\qquad(\Rightarrow\neg_3\neg_3^*)\ \frac{\neg_3A,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,\neg_3\neg_3A}$$

$$(\neg_4\neg_4)\ \frac{\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,\neg_4\neg_4A}$$

$$(\neg_5\neg_5\Rightarrow)\ \frac{A,\Gamma\Rightarrow\Delta}{\neg_5\neg_5A,\Gamma\Rightarrow\Delta}\qquad(\neg_5\neg_5\Rightarrow^*)\ \frac{\Gamma\Rightarrow\Delta,\neg_5A}{\neg_5\neg_5A,\Gamma\Rightarrow\Delta}\qquad(\Rightarrow\neg_5\neg_5)\ \frac{\Gamma\Rightarrow\Delta,A\quad\neg_5A,\Pi\Rightarrow\Sigma}{\Gamma,\Pi\Rightarrow\Delta,\Sigma,\neg_5\neg_5A}$$

$$(\neg_6\neg_6\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,A\quad\Pi\Rightarrow\Sigma,\neg_6A}{\neg_6\neg_6A,\Gamma,\Pi\Rightarrow\Delta,\Sigma}\qquad(\neg_6\neg_6\Rightarrow^*)\ \frac{A,\Gamma\Rightarrow\Delta,\quad\neg_6A,\Pi\Rightarrow\Sigma}{\neg_6\neg_6A,\Gamma,\Pi\Rightarrow\Delta,\Sigma}$$

$$(\Rightarrow\neg_6\neg_6)\ \frac{\Gamma\Rightarrow\Delta,A\quad\neg_6A,\Pi\Rightarrow\Sigma}{\Gamma,\Pi\Rightarrow\Delta,\Sigma,\neg_6\neg_6A}\qquad(\Rightarrow\neg_6\neg_6^*)\ \frac{\Gamma\Rightarrow\Delta,\neg_6A\quad A,\Pi\Rightarrow\Sigma}{\Gamma,\Pi\Rightarrow\Delta,\Sigma,\neg_6\neg_6A}$$

$$(\neg_7\neg_7\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,\neg_7A}{\neg_7\neg_7A,\Gamma\Rightarrow\Delta}\qquad(\Rightarrow\neg_7\neg_7)\ \frac{\neg_7A,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta,\neg_7\neg_7A}$$

[15] Kooi and Tamminga postulate weakening, although they notice that this rule is not necessary due to the axiom $A,\Gamma\Rightarrow\Delta,A$. This rule would be necessary if this axiom had the following form: $A\Rightarrow A$.

$$(\neg_8\neg_8 \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \quad \Pi \Rightarrow \Sigma, \neg_8 A}{\neg_8\neg_8 A, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (\Rightarrow \neg_8\neg_8)\ \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_8\neg_8 A} \qquad (\Rightarrow \neg_8\neg_8^*)\ \frac{\neg_8 A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_8\neg_8 A}$$

$$(\neg_9\neg_9 \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta}{\neg_9\neg_9 A, \Gamma \Rightarrow \Delta} \qquad (\neg_9\neg_9 \Rightarrow^*)\ \frac{\neg_9 A, \Gamma \Rightarrow \Delta}{\neg_9\neg_9 A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_9\neg_9)\ \frac{\Gamma \Rightarrow \Delta, A \quad \Pi \Rightarrow \Sigma, \neg_9 A}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \neg_9\neg_9 A}$$

$$(\neg_{10}\neg_{10} \Rightarrow)\ \frac{\neg_{10} A, \Gamma \Rightarrow \Delta}{\neg_{10}\neg_{10} A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_{10}\neg_{10})\ \frac{\Gamma \Rightarrow \Delta, \neg_{10} A}{\Gamma \Rightarrow \Delta, \neg_{10}\neg_{10} A}$$

$$(\neg_{11}\neg_{11} \Rightarrow)\ \frac{\neg_{11} A, \Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma, A}{\neg_{11}\neg_{11} A, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (\neg_{11}\neg_{11} \Rightarrow^*)\ \frac{A, \Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma, \neg_{11} A}{\neg_{11}\neg_{11} A, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(\Rightarrow \neg_{11}\neg_{11})\ \frac{A, \Gamma \Rightarrow \Delta \quad \neg_{11} A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \neg_{11}\neg_{11} A} \qquad (\Rightarrow \neg_{11}\neg_{11}^*)\ \frac{\Gamma \Rightarrow \Delta, A \quad \Pi \Rightarrow \Sigma, \neg_{11} A}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \neg_{11}\neg_{11} A}$$

$$(\neg_{12}\neg_{12} \Rightarrow)\ \frac{\neg_{12} A, \Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma, A}{\neg_{12}\neg_{12} A, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (\Rightarrow \neg_{12}\neg_{12})\ \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_{12}\neg_{12} A} \qquad (\Rightarrow \neg_{12}\neg_{12})\ \frac{\Gamma \Rightarrow \Delta, \neg_{12} A}{\Gamma \Rightarrow \Delta, \neg_{12}\neg_{12} A}$$

$$(\neg_{13}\neg_{13})\ \frac{\Gamma \Rightarrow \Delta}{\neg_{13}\neg_{13} A, \Gamma \Rightarrow \Delta}$$

$$(\neg_{14}\neg_{14} \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A}{\neg_{14}\neg_{14} A, \Gamma \Rightarrow \Delta} \qquad (\neg_{14}\neg_{14} \Rightarrow^*)\ \frac{\neg_{14} A, \Gamma \Rightarrow \Delta}{\neg_{14}\neg_{14} A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_{14}\neg_{14})\ \frac{\Gamma \Rightarrow \Delta, \neg_{14} A \quad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \neg_{14}\neg_{14} A}$$

$$(\neg_{15}\neg_{15} \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A}{\neg_{15}\neg_{15} A, \Gamma \Rightarrow \Delta} \qquad (\neg_{15}\neg_{15} \Rightarrow^*)\ \frac{\Gamma \Rightarrow \Delta, \neg_{15} A}{\neg_{15}\neg_{15} A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_{15}\neg_{15})\ \frac{A, \Gamma \Rightarrow \Delta \quad \neg_{15} A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \neg_{15}\neg_{15} A}$$

$$(\neg_{16}\neg_{16} \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A}{\neg_{16}\neg_{16} A, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \neg_{16}\neg_{16})\ \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg_{16}\neg_{16} A}$$

Corollary 129 (Adequacy)**.** *Let* $\mathbf{L}$ *be* $\mathbf{FDE}_{\neg}^{i(\circledcirc)_m}$, *for any* $1 \leqslant i \leqslant 16$. *For any finite multisets of formulas* Γ *and* Δ, $\Gamma \models_{\mathbf{L}} \Delta$ *iff* $\vdash_{\mathbf{L}} \Gamma \Rightarrow \Delta$.

Proof. For the case $i = 1$ this theorem is proven in [97, Theorems 1, 2]. The other cases are considered similarly; when we consider modal extensions of these four-valued logics, we will give some tips for the completeness proof of the propositional case as well (see Theorem 143). □

The sequent rules for an n-ary connective $\circledcirc$ can be formulated in Segerberg's notation as follows:

$(\Rightarrow^{\mathbf{FDE}}_{\langle \mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle 1} \circledcirc)$

$$\frac{\{\Gamma \Rightarrow \Delta, A_i\}^\dagger\ \{\neg A_i, \Gamma \Rightarrow \Delta\}^\dagger \quad \{\Gamma \Rightarrow \Delta, A_j\}^\S\ \{\Gamma \Rightarrow \Delta, \neg A_j\}^\S \quad \{A_k, \Gamma \Rightarrow \Delta\}^\ddagger\ \{\neg A_k, \Gamma \Rightarrow \Delta\}^\ddagger \quad \{A_l, \Gamma \Rightarrow \Delta\}^*\ \{\Gamma \Rightarrow \Delta, \neg A_l\}^*}{\Gamma \Rightarrow \Delta, \circledcirc(A_1, \ldots, A_n)}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$(\neg\circledcirc \Rightarrow^{\mathbf{FDE}}_{\langle \mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle 1})$

$$\frac{\{\Gamma \Rightarrow \Delta, A_i\}^\dagger\ \{\neg A_i, \Gamma \Rightarrow \Delta\}^\dagger \quad \{\Gamma \Rightarrow \Delta, A_j\}^\S\ \{\Gamma \Rightarrow \Delta, \neg A_j\}^\S \quad \{A_k, \Gamma \Rightarrow \Delta\}^\ddagger\ \{\neg A_k, \Gamma \Rightarrow \Delta\}^\ddagger \quad \{A_l, \Gamma \Rightarrow \Delta\}^*\ \{\Gamma \Rightarrow \Delta, \neg A_l\}^*}{\neg\circledcirc(A_1, \ldots, A_n), \Gamma \Rightarrow \Delta}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$(\Rightarrow^{\mathbf{FDE}}_{\langle \mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle b} \circledcirc)$

$$\frac{\{\Gamma \Rightarrow \Delta, A_i\}^\dagger\ \{\neg A_i, \Gamma \Rightarrow \Delta\}^\dagger \quad \{\Gamma \Rightarrow \Delta, A_j\}^\S\ \{\Gamma \Rightarrow \Delta, \neg A_j\}^\S \quad \{A_k, \Gamma \Rightarrow \Delta\}^\ddagger\ \{\neg A_k, \Gamma \Rightarrow \Delta\}^\ddagger \quad \{A_l, \Gamma \Rightarrow \Delta\}^*\ \{\Gamma \Rightarrow \Delta, \neg A_l\}^*}{\Gamma \Rightarrow \Delta, \circledcirc(A_1, \ldots, A_n)}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$(\Rightarrow^{\mathbf{FDE}}_{\langle \mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle b} \neg\circledcirc)$

$$\frac{\{\Gamma \Rightarrow \Delta, A_i\}^\dagger\ \{\neg A_i, \Gamma \Rightarrow \Delta\}^\dagger \quad \{\Gamma \Rightarrow \Delta, A_j\}^\S\ \{\Gamma \Rightarrow \Delta, \neg A_j\}^\S \quad \{A_k, \Gamma \Rightarrow \Delta\}^\ddagger\ \{\neg A_k, \Gamma \Rightarrow \Delta\}^\ddagger \quad \{A_l, \Gamma \Rightarrow \Delta\}^*\ \{\Gamma \Rightarrow \Delta, \neg A_l\}^*}{\Gamma \Rightarrow \Delta, \neg\circledcirc(A_1, \ldots, A_n)}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$(\circledcirc \Rightarrow^{\mathbf{FDE}}_{\langle \mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle n})$

$$\frac{\{\Gamma \Rightarrow \Delta, A_i\}^{\dagger}\ \{\neg A_i, \Gamma \Rightarrow \Delta\}^{\dagger}\quad \{\Gamma \Rightarrow \Delta, A_j\}^{\S}\ \{\Gamma \Rightarrow \Delta, \neg A_j\}^{\S}\quad \{A_k, \Gamma \Rightarrow \Delta\}^{\ddagger}\ \{\neg A_k, \Gamma \Rightarrow \Delta\}^{\ddagger}\quad \{A_l, \Gamma \Rightarrow \Delta\}^{*}\ \{\Gamma \Rightarrow \Delta, \neg A_l\}^{*}}{\circledcirc(A_1, \ldots, A_n), \Gamma \Rightarrow \Delta}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$(\neg\circledcirc \Rightarrow^{\mathbf{FDE}}_{\langle \mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle n})$

$$\frac{\{\Gamma \Rightarrow \Delta, A_i\}^{\dagger}\ \{\neg A_i, \Gamma \Rightarrow \Delta\}^{\dagger}\quad \{\Gamma \Rightarrow \Delta, A_j\}^{\S}\ \{\Gamma \Rightarrow \Delta, \neg A_j\}^{\S}\quad \{A_k, \Gamma \Rightarrow \Delta\}^{\ddagger}\ \{\neg A_k, \Gamma \Rightarrow \Delta\}^{\ddagger}\quad \{A_l, \Gamma \Rightarrow \Delta\}^{*}\ \{\Gamma \Rightarrow \Delta, \neg A_l\}^{*}}{\neg\circledcirc(A_1, \ldots, A_n), \Gamma \Rightarrow \Delta}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$(\circledcirc \Rightarrow^{\mathbf{FDE}}_{\langle \mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle 0})$

$$\frac{\{\Gamma \Rightarrow \Delta, A_i\}^{\dagger}\ \{\neg A_i, \Gamma \Rightarrow \Delta\}^{\dagger}\quad \{\Gamma \Rightarrow \Delta, A_j\}^{\S}\ \{\Gamma \Rightarrow \Delta, \neg A_j\}^{\S}\quad \{A_k, \Gamma \Rightarrow \Delta\}^{\ddagger}\ \{\neg A_k, \Gamma \Rightarrow \Delta\}^{\ddagger}\quad \{A_l, \Gamma \Rightarrow \Delta\}^{*}\ \{\Gamma \Rightarrow \Delta, \neg A_l\}^{*}}{\circledcirc(A_1, \ldots, A_n), \Gamma \Rightarrow \Delta}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$(\Rightarrow^{\mathbf{FDE}}_{\langle \mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle 0} \neg\circledcirc)$

$$\frac{\{\Gamma \Rightarrow \Delta, A_i\}^{\dagger}\ \{\neg A_i, \Gamma \Rightarrow \Delta\}^{\dagger}\quad \{\Gamma \Rightarrow \Delta, A_j\}^{\S}\ \{\Gamma \Rightarrow \Delta, \neg A_j\}^{\S}\quad \{A_k, \Gamma \Rightarrow \Delta\}^{\ddagger}\ \{\neg A_k, \Gamma \Rightarrow \Delta\}^{\ddagger}\quad \{A_l, \Gamma \Rightarrow \Delta\}^{*}\ \{\Gamma \Rightarrow \Delta, \neg A_l\}^{*}}{\Gamma \Rightarrow \Delta, \neg\circledcirc(A_1, \ldots, A_n)}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

Let us present a constructive cut admissibility proof for $\mathbf{FDE}^{(\circledcirc)_m}_{\neg}$. We use the strategy by Metcalfe, Olivetti, and Gabbay [123] which we have already applied in Section 2.3.2. The notions of *principal formula*, *side formulas*, *parametric formulas*, the length $\mathfrak{l}(\mathfrak{D})$ of a derivation $\mathfrak{D}$, and the cut rank $\mathfrak{r}(\mathfrak{D})$ of a derivation $\mathfrak{D}$ are given in Section 2.3.2. The complexity $\mathfrak{c}(A)$ of A is defined as a degree $d(A)$ of A introduced in Definition 96.

Lemma 130 (Right reduction)**.** *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be derivations such that:*

(1) $\mathfrak{D}_1$ *is a derivation of* $\Gamma \Rightarrow \Delta, A$,

(2) $\mathfrak{D}_2$ *is a derivation of* $A^a, \Theta \Rightarrow \Lambda$,

(3) $\mathfrak{r}(\mathfrak{D}_1) \leq \mathfrak{c}(A)$ *and* $\mathfrak{r}(\mathfrak{D}_2) \leq \mathfrak{c}(A)$,

(4) A *is the principal formula of a logical rule in* $\mathfrak{D}_1$.

Then we can construct a derivation $\mathfrak{D}_0$ of $\Gamma^a, \Theta \Rightarrow \Lambda, \Delta^a$ such that $\mathfrak{r}(\mathfrak{D}_0) \leq \mathfrak{c}(A)$.

Proof. By induction on $\mathfrak{l}(\mathfrak{D}_2)$.

Basic case. Similar to the basic case from Lemma 105.

We have the following application of (Cut) on the left and its obvious transformation on the right:

Inductive case. We have different cases depending on the last rule applied in $\mathfrak{D}_2$.

Case 1. The rule of the last application in $\mathfrak{D}_2$ is $(\neg\neg \Rightarrow)$. Similar to the case with this rule from Lemma 105.

Case 2. The rule of the last application in $\mathfrak{D}_2$ is $(\neg\circledcirc \Rightarrow^{\mathbf{FDE}}_{\langle \mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle 1})$. *Subcase 2.1.* A is principal in $\mathfrak{D}_2$ and $A = \neg\circledcirc(B_1, \ldots, B_n) = \neg\circledcirc(\vec{B})$. Let us introduce the following abbreviations:

$\mathfrak{A}_1 = \{A^a, \Theta \Rightarrow \Lambda, B_i\}^{\dagger}\quad \{A^a, \neg B_i, \Theta \Rightarrow \Lambda\}^{\dagger}$
$\mathfrak{A}_2 = \{A^a, \Theta \Rightarrow \Lambda, B_j\}^{\S}\quad \{A^a, \Theta \Rightarrow \Lambda, \neg B_j\}^{\S}$
$\mathfrak{A}_3 = \{A^a, B_k, \Theta \Rightarrow \Lambda\}^{\ddagger}\quad \{A^a, \neg B_k, \Theta \Rightarrow \Lambda\}^{\ddagger}$
$\mathfrak{A}_4 = \{A^a, B_l, \Theta \Rightarrow \Lambda\}^{*}\quad \{A^a, \Theta \Rightarrow \Lambda, \neg B_l\}^{*}$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

The last inference of $\mathfrak{D}_2$ is as follows.

$$\frac{\mathfrak{A}_1 \quad \mathfrak{A}_2 \quad \mathfrak{A}_3 \quad \mathfrak{A}_4}{\neg\circledcirc(B_1, \ldots, B_n)^{a+1}, \Theta \Rightarrow \Lambda}$$

The last inference of $\mathfrak{D}_1$ is as follows (the rule $(\Rightarrow^{\mathbf{FDE}}_{\langle \mathfrak{t}',\mathfrak{b}',\mathfrak{n}',\mathfrak{f}'\rangle b} \neg\circledcirc)$ is applied):

$$\frac{\{\Gamma\Rightarrow\Delta,B_{i'}\}^{\dagger}\ \{\neg B_{i'},\Gamma\Rightarrow\Delta\}^{\dagger}\quad \{\Gamma\Rightarrow\Delta,B_{j'}\}^{\S}\ \{\Gamma\Rightarrow\Delta,\neg B_{j'}\}^{\S}\quad \{B_{k'},\Gamma\Rightarrow\Delta\}^{\ddagger}\ \{\neg B_{k'},\Gamma\Rightarrow\Delta\}^{\ddagger}\quad \{B_{l'},\Gamma\Rightarrow\Delta\}^{*}\ \{\Gamma\Rightarrow\Delta,\neg B_{l'}\}^{*}}{\Gamma\Rightarrow\Delta,\neg\circledcirc(B_1,\ldots,B_n)}$$

† for each $i'\in\mathfrak{t}'$, § for each $j'\in\mathfrak{b}'$, ‡ for each $k'\in\mathfrak{n}'$, * for each $l'\in\mathfrak{f}'$.

We should obtain $\Gamma^{a+1},\Theta\Rightarrow\Delta^{a+1},\Lambda$. Recall that all the formulas

$$B_i,\neg B_i,B_j,\neg B_j,B_k,\neg B_k,B_l,\neg B_l,B_{i'},\neg B_{i'},B_{j'},\neg B_{j'},B_{k'},\neg B_{k'},B_{l'},\neg B_{l'}$$

are either subformulas or negations of subformulas of $\circledcirc(\vec{B})=\circledcirc(B_1,\ldots,B_n)$. We need to have more details about the other rules which are present in the system. Let us observe that this case can be reformulated in the following way (the last step of $\mathfrak{D}_1$ is on the left, the last step of $\mathfrak{D}_2$ is on the right):

$$\frac{X_1\ldots X_n}{\Gamma\Rightarrow\Delta,A}\qquad\frac{Y_1\ldots Y_n}{A^{a+1},\Theta\Rightarrow\Lambda}$$

where $X_1,\ldots,X_n$ and $Y_1,\ldots,Y_n$ are abbreviations for the following derivations, for any $t,u\in\{1,\ldots,n\}$.

$$X_t=\begin{cases}\left[\begin{array}{l}\Gamma\Rightarrow\Delta,B_t\\ \neg B_t,\Gamma\Rightarrow\Delta\end{array}\right. & \textit{iff}\quad t\in\mathfrak{t}';\\[2ex] \left[\begin{array}{l}\Gamma\Rightarrow\Delta,B_t\\ \Gamma\Rightarrow\Delta,\neg B_t\end{array}\right. & \textit{iff}\quad t\in\mathfrak{b}';\\[2ex] \left[\begin{array}{l}B_t,\Gamma\Rightarrow\Delta\\ \neg B_t,\Gamma\Rightarrow\Delta\end{array}\right. & \textit{iff}\quad t\in\mathfrak{n}';\\[2ex] \left[\begin{array}{l}B_t,\Gamma\Rightarrow\Delta\\ \Gamma\Rightarrow\Delta,\neg B_t\end{array}\right. & \textit{iff}\quad t\in\mathfrak{f}';\end{cases}\qquad Y_u=\begin{cases}\left[\begin{array}{l}A^a,\Theta\Rightarrow\Lambda,B_u\\ A^a,\neg B_u,\Theta\Rightarrow\Lambda\end{array}\right. & \textit{iff}\quad u\in\mathfrak{t};\\[2ex] \left[\begin{array}{l}A^a,\Theta\Rightarrow\Lambda,B_u\\ A^a,\Theta\Rightarrow\Lambda,\neg B_u\end{array}\right. & \textit{iff}\quad u\in\mathfrak{b};\\[2ex] \left[\begin{array}{l}A^a,B_u,\Theta\Rightarrow\Lambda\\ A^a,\neg B_u,\Theta\Rightarrow\Lambda\end{array}\right. & \textit{iff}\quad u\in\mathfrak{n};\\[2ex] \left[\begin{array}{l}A^a,B_u,\Theta\Rightarrow\Lambda\\ A^a,\Theta\Rightarrow\Lambda,\neg B_u\end{array}\right. & \textit{iff}\quad u\in\mathfrak{f};\end{cases}$$

As follows from these equalities and the soundness of our natural deduction systems, there is $s\in\{1,\ldots,n\}$ such that $X_s\neq Y_s$. Then the following combinations are possible (we present them in the form of ordered pairs $\langle X_s,Y_s\rangle$):

$$\mathcal{C}_1=\left\langle\left[\begin{array}{l}\Gamma\Rightarrow\Delta,B_s\\ \neg B_s,\Gamma\Rightarrow\Delta\end{array}\right.,\left[\begin{array}{l}A^a,\Theta\Rightarrow\Lambda,B_s\\ A^a,\Theta\Rightarrow\Lambda,\neg B_s\end{array}\right.\right\rangle\quad \mathcal{C}_2=\left\langle\left[\begin{array}{l}\Gamma\Rightarrow\Delta,B_s\\ \neg B_s,\Gamma\Rightarrow\Delta\end{array}\right.,\left[\begin{array}{l}A^a,B_s,\Theta\Rightarrow\Lambda\\ A^a,\neg B_s,\Theta\Rightarrow\Lambda\end{array}\right.\right\rangle$$

$$\mathcal{C}_3=\left\langle\left[\begin{array}{l}\Gamma\Rightarrow\Delta,B_s\\ \neg B_s,\Gamma\Rightarrow\Delta\end{array}\right.,\left[\begin{array}{l}A^a,B_s,\Theta\Rightarrow\Lambda\\ A^a,\Theta\Rightarrow\Lambda,\neg B_s\end{array}\right.\right\rangle\quad \mathcal{C}_4=\left\langle\left[\begin{array}{l}\Gamma\Rightarrow\Delta,B_s\\ \Gamma\Rightarrow\Delta,\neg B_s\end{array}\right.,\left[\begin{array}{l}A^a,\Theta\Rightarrow\Lambda,B_s\\ A^a,\neg B_s,\Theta\Rightarrow\Lambda\end{array}\right.\right\rangle$$

$$\mathcal{C}_5=\left\langle\left[\begin{array}{l}\Gamma\Rightarrow\Delta,B_s\\ \Gamma\Rightarrow\Delta,\neg B_s\end{array}\right.,\left[\begin{array}{l}A^a,B_s,\Theta\Rightarrow\Lambda\\ A^a,\neg B_s,\Theta\Rightarrow\Lambda\end{array}\right.\right\rangle\quad \mathcal{C}_6=\left\langle\left[\begin{array}{l}\Gamma\Rightarrow\Delta,B_s\\ \Gamma\Rightarrow\Delta,\neg B_s\end{array}\right.,\left[\begin{array}{l}A^a,B_s,\Theta\Rightarrow\Lambda\\ A^a,\Theta\Rightarrow\Lambda,\neg B_s\end{array}\right.\right\rangle$$

$$\mathcal{C}_7=\left\langle\left[\begin{array}{l}B_s,\Gamma\Rightarrow\Delta\\ \neg B_s,\Gamma\Rightarrow\Delta\end{array}\right.,\left[\begin{array}{l}A^a,\Theta\Rightarrow\Lambda,B_s\\ A^a,\neg B_s,\Theta\Rightarrow\Lambda\end{array}\right.\right\rangle\quad \mathcal{C}_8=\left\langle\left[\begin{array}{l}B_s,\Gamma\Rightarrow\Delta\\ \neg B_s,\Gamma\Rightarrow\Delta\end{array}\right.,\left[\begin{array}{l}A^a,\Theta\Rightarrow\Lambda,B_s\\ A^a,\Theta\Rightarrow\Lambda,\neg B_s\end{array}\right.\right\rangle$$

$$\mathcal{C}_9=\left\langle\left[\begin{array}{l}B_s,\Gamma\Rightarrow\Delta\\ \neg B_s,\Gamma\Rightarrow\Delta\end{array}\right.,\left[\begin{array}{l}A^a,B_s,\Theta\Rightarrow\Lambda\\ A^a,\Theta\Rightarrow\Lambda,\neg B_s\end{array}\right.\right\rangle\quad \mathcal{C}_{10}=\left\langle\left[\begin{array}{l}B_s,\Gamma\Rightarrow\Delta\\ \Gamma\Rightarrow\Delta,\neg B_s\end{array}\right.,\left[\begin{array}{l}A^a,\Theta\Rightarrow\Lambda,B_s\\ A^a,\neg B_s,\Theta\Rightarrow\Lambda\end{array}\right.\right\rangle$$

$$\mathcal{C}_{11}=\left\langle\left[\begin{array}{l}B_s,\Gamma\Rightarrow\Delta\\ \Gamma\Rightarrow\Delta,\neg B_s\end{array}\right.,\left[\begin{array}{l}A^a,\Theta\Rightarrow\Lambda,B_s\\ A^a,\Theta\Rightarrow\Lambda,\neg B_s\end{array}\right.\right\rangle\quad \mathcal{C}_{12}=\left\langle\left[\begin{array}{l}B_s,\Gamma\Rightarrow\Delta\\ \Gamma\Rightarrow\Delta,\neg B_s\end{array}\right.,\left[\begin{array}{l}A^a,B_s,\Theta\Rightarrow\Lambda\\ A^a,\neg B_s,\Theta\Rightarrow\Lambda\end{array}\right.\right\rangle$$

Let us consider the case $\mathcal{C}_1$. The derivation $\mathfrak{D}_1$ is as follows:

$$\frac{X_1\ldots X_{l-1}\quad \Gamma\Rightarrow\Delta,B_s\quad \neg B_s,\Gamma\Rightarrow\Delta\quad X_{l+1}\ldots X_n}{\Gamma\Rightarrow\Delta,A}$$

The derivation $\mathfrak{D}_2$ is as follows:

$$\frac{Y_1 \dots Y_{l-1} \quad A^a, \Theta \Rightarrow \Lambda, B_s \quad A^a, \Theta \Rightarrow \Lambda, \neg B_s \quad Y_{l+1} \dots Y_n}{A^{a+1}, \Theta \Rightarrow \Lambda}$$

We should obtain $\Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda$. Using the induction hypothesis, applying cut to the formulas of lower complexity[16], and using structural rules, we get the required result:

$$\cfrac{\cfrac{\Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda, \neg B_s \qquad \neg B_s, \Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda}{\Gamma^{2(a+1)}, \Theta^2 \Rightarrow \Delta^{2(a+1)}, \Lambda^2}}{\Gamma^{a+1}, \Theta \Rightarrow \Delta^{a+1}, \Lambda}$$

The other cases are treated similarly. □

Lemma 131 (Left reduction). *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be derivations such that*:

(1) $\mathfrak{D}_1$ *is a derivation of* $\Gamma \Rightarrow \Delta, A^i$,

(2) $\mathfrak{D}_2$ *is a derivation of* $A, \Theta \Rightarrow \Lambda$,

(3) $\mathfrak{r}(\mathfrak{D}_1) \leq \mathfrak{c}(A)$ *and* $\mathfrak{r}(\mathfrak{D}_2) \leq \mathfrak{c}(A)$.

Then we can construct a derivation $\mathfrak{D}_0$ of $\Gamma, \Theta^i \Rightarrow \Lambda^i, \Delta$ such that $\mathfrak{r}(\mathfrak{D}_0) \leq \mathfrak{c}(A)$.

Proof. The proof is by induction on $\mathfrak{l}(\mathfrak{D}_1)$. Similarly to Lemma 130. □

Theorem 132 (Constructive elimination of cuts). *Let* $\mathbf{L} = \mathbf{FDE}_{\neg}^{i(\circledcirc)_m}$, $1 \leqslant i \leqslant 16$. *If a derivation $\mathfrak{D}$ in $\mathbb{SC}_\mathbf{L}$ has an application of* (Cut), *then it can be transformed into a cut-free derivation $\mathfrak{D}'$.*

Proof. Assume that a derivation $\mathfrak{D}$ in $\mathbb{SC}_\mathbf{L}$ has at least one application of (Cut), i.e., $\mathfrak{r}(\mathfrak{D}) > 0$. The proof proceeds by the double induction on $\langle \mathfrak{r}(\mathfrak{D}), \mathfrak{nr}(\mathfrak{D}) \rangle$, where $\mathfrak{nr}(\mathfrak{D})$ is the number of applications of (Cut) in $\mathfrak{D}$. Consider an uppermost application of (Cut) in $\mathfrak{D}$ with cut rank $\mathfrak{r}(\mathfrak{D})$. We apply Lemma 131 to its premises and decrease either $\mathfrak{r}(\mathfrak{D})$ or $\mathfrak{nr}(\mathfrak{D})$. Then we can use the inductive hypothesis. □

3.6.4 Natural deduction for four-valued logics

Let us present now natural deduction systems for the four-valued logics $\mathbf{FDE}_{\neg}^{i(\circledcirc)_m}$, $1 \leqslant i \leqslant 16$, on the basis of Kooi and Tamminga's sequent calculi.

Let us adapt Kooi and Tamminga's notation for the case of natural deduction.

Definition 133. For any formulas A and B, $z \in \{1, b, n, 0\}$, $A|_z^+(B)$, $A||_z^+(B)$, $A|_z^-(B)$, and $A||_z^-(B)$ are defined as follows:

$$A|_z^+(B) = \begin{cases} \begin{matrix}[A]\\ \mathfrak{D}\\ B\end{matrix} & \text{if } z \in \{n, 0\}, \\[2ex] \begin{matrix}\mathfrak{E}\\ A\end{matrix} & \text{otherwise} \end{cases} \qquad A|_z^-(B) = \begin{cases} \begin{matrix}[\neg A]\\ \mathfrak{D}\\ B\end{matrix} & \text{if } z \in \{n, 1\}, \\[2ex] \begin{matrix}\mathfrak{E}\\ \neg A\end{matrix} & \text{otherwise} \end{cases}$$

$$A||_z^+(B) = \begin{cases} \begin{matrix}\mathfrak{E}\\ A\end{matrix} & \text{if } z \in \{n, 0\}, \\[2ex] \begin{matrix}[A]\\ \mathfrak{D}\\ B\end{matrix} & \text{otherwise} \end{cases} \qquad A||_z^-(B) = \begin{cases} \begin{matrix}\mathfrak{E}\\ \neg A\end{matrix} & \text{if } z \in \{n, 1\}, \\[2ex] \begin{matrix}[\neg A]\\ \mathfrak{D}\\ B\end{matrix} & \text{otherwise} \end{cases}$$

[16] Recall that due to our definition of the degree of formula, the complexity of $\circledcirc(B_1, \dots, B_n)$ is lower than the degree of $\neg B_s$, even if $n = s = 1$.

Natural deduction versions of Kooi and Tamminga's sequent rules for four-valued n-ary connectives (nd stands for natural deduction) are as follows:

$$R^{\circledcirc + nd}_{x_1,\ldots,x_n,y}\ \frac{\circledcirc(A_1,\ldots,A_n)||^{+}_{y}(B) \quad A_1|^{+}_{x_1}(B) \quad A_1|^{-}_{x_1}(B) \quad \ldots \quad A_n|^{+}_{x_n}(B) \quad A_n|^{-}_{x_n}(B)}{B}$$

$$R^{\circledcirc - nd}_{x_1,\ldots,x_n,y}\ \frac{\circledcirc(A_1,\ldots,A_n)||^{-}_{y}(B) \quad A_1|^{+}_{x_1}(B) \quad A_1|^{-}_{x_1}(B) \quad \ldots \quad A_n|^{+}_{x_n}(B) \quad A_n|^{-}_{x_n}(B)}{B}$$

The rules for negation were formulated above, as general introduction and elimination rules.

These rules can be formulated in Segerberg's notation too, in the same style which we used for three-valued logics.

Notation 134. Consider the set of natural numbers $\mathfrak{s} = \{1,\ldots,n\}$. By a 4-partitioning of $\mathfrak{s}$ we mean an ordered quadruple $\langle I, J, K, L\rangle$ such that $I \cup J \cup K \cup L = \mathfrak{s}$ and $I \cap J \cap K \cap L = \emptyset$. In what follows, we are going to consider a partitioning of the following type: $\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f}\rangle$, where $\mathfrak{t} = \{i \in \mathfrak{s} \mid v(A_i) = 1, A_i \in \mathscr{F}_{(\circledcirc)m}\}$, $\mathfrak{b} = \{j \in \mathfrak{s} \mid v(A_j) = b, A_j \in \mathscr{F}_{(\circledcirc)m}\}$, $\mathfrak{n} = \{k \in \mathfrak{s} \mid v(A_k) = n, A_k \in \mathscr{F}_{(\circledcirc)m}\}$, and $\mathfrak{f} = \{l \in \mathfrak{s} \mid v(A_l) = 0, A_l \in \mathscr{F}_{(\circledcirc)m}\}$.

Notation 135. The expression $f_{\circledcirc}(x_1,\ldots,x_n) = y$, where $x_1,\ldots,x_n, y \in \{1, n, b, 0\}$, means that if $v(A_1) = x_1$, $\ldots$, $v(A_n) = x_n$, then $v(\circledcirc(A_1,\ldots,A_n)) = y$, for each valuation v and all formulas $A_1,\ldots,A_n$. The expression $f_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f}\rangle\big) = x$, where $x \in \{1, b, n, 0\}$, means that if $v(A_i) = 1$ (for each $i \in \mathfrak{t}$), $v(A_j) = b$ (for each $j \in \mathfrak{b}$), $v(A_k) = n$ (for each $k \in \mathfrak{n}$), and $v(A_l) = 0$ (for each $l \in \mathfrak{f}$), then $v(\circledcirc(A_1,\ldots,A_n)) = x$, for each valuation v.

$$R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f}\rangle 1\big)^{a,b,c,d,e}\ \frac{\begin{array}{ccccccccc} [\circledcirc(A_1,\ldots,A_n)]^a & & [\neg A_i]^b & & & [A^{\ddagger}_k]^c & [\neg A^{\ddagger}_k]^d & [A^{*}_l]^e & \\ \mathfrak{D}_1 & \mathfrak{D}^{\dagger}_2 & \mathfrak{D}^{\dagger}_3 & \mathfrak{D}^{\S}_4 & \mathfrak{D}^{\S}_5 & \mathfrak{D}^{\ddagger}_6 & \mathfrak{D}^{\ddagger}_7 & \mathfrak{D}^{*}_8 & \mathfrak{D}^{*}_9 \\ B & A^{\dagger}_i & B & A^{\S}_j & \neg A^{\S}_j & B & B & B & \neg A^{*}_l \end{array}}{B}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$$R^{-}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f}\rangle 1\big)^{b,c,d,e}\ \frac{\begin{array}{ccccccccc} & & [\neg A_i]^b & & & [A^{\ddagger}_k]^c & [\neg A^{\ddagger}_k]^d & [A^{*}_l]^e & \\ \mathfrak{D}_1 & \mathfrak{D}^{\dagger}_2 & \mathfrak{D}^{\dagger}_3 & \mathfrak{D}^{\S}_4 & \mathfrak{D}^{\S}_5 & \mathfrak{D}^{\ddagger}_6 & \mathfrak{D}^{\ddagger}_7 & \mathfrak{D}^{*}_8 & \mathfrak{D}^{*}_9 \\ \neg\circledcirc(A_1,\ldots,A_n) & A^{\dagger}_i & B & A^{\S}_j & \neg A^{\S}_j & B & B & B & \neg A^{*}_l \end{array}}{B}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$$R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f}\rangle b\big)^{a,b,c,d,e}\ \frac{\begin{array}{ccccccccc} [\circledcirc(A_1,\ldots,A_n)]^a & & [\neg A_i]^b & & & [A^{\ddagger}_k]^c & [\neg A^{\ddagger}_k]^d & [A^{*}_l]^e & \\ \mathfrak{D}_1 & \mathfrak{D}^{\dagger}_2 & \mathfrak{D}^{\dagger}_3 & \mathfrak{D}^{\S}_4 & \mathfrak{D}^{\S}_5 & \mathfrak{D}^{\ddagger}_6 & \mathfrak{D}^{\ddagger}_7 & \mathfrak{D}^{*}_8 & \mathfrak{D}^{*}_9 \\ B & A^{\dagger}_i & B & A^{\S}_j & \neg A^{\S}_j & B & B & B & \neg A^{*}_l \end{array}}{B}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$$R^{-}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f}\rangle b\big)^{a,b,c,d,e}\ \frac{\begin{array}{ccccccccc} [\neg\circledcirc(A_1,\ldots,A_n)]^a & & [\neg A_i]^b & & & [A^{\ddagger}_k]^c & [\neg A^{\ddagger}_k]^d & [A^{*}_l]^e & \\ \mathfrak{D}_1 & \mathfrak{D}^{\dagger}_2 & \mathfrak{D}^{\dagger}_3 & \mathfrak{D}^{\S}_4 & \mathfrak{D}^{\S}_5 & \mathfrak{D}^{\ddagger}_6 & \mathfrak{D}^{\ddagger}_7 & \mathfrak{D}^{*}_8 & \mathfrak{D}^{*}_9 \\ B & A^{\dagger}_i & B & A^{\S}_j & \neg A^{\S}_j & B & B & B & \neg A^{*}_l \end{array}}{B}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$$R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f}\rangle n\big)^{b,c,d,e}\ \frac{\begin{array}{ccccccccc} & & [\neg A_i]^b & & & [A^{\ddagger}_k]^c & [\neg A^{\ddagger}_k]^d & [A^{*}_l]^e & \\ \mathfrak{D}_1 & \mathfrak{D}^{\dagger}_2 & \mathfrak{D}^{\dagger}_3 & \mathfrak{D}^{\S}_4 & \mathfrak{D}^{\S}_5 & \mathfrak{D}^{\ddagger}_6 & \mathfrak{D}^{\ddagger}_7 & \mathfrak{D}^{*}_8 & \mathfrak{D}^{*}_9 \\ \circledcirc(A_1,\ldots,A_n) & A^{\dagger}_i & B & A^{\S}_j & \neg A^{\S}_j & B & B & B & \neg A^{*}_l \end{array}}{B}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$$R^{\neg}_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle n\big)^{b,c,d,e}\ \frac{\begin{array}{c}\\ \mathfrak{D}_1\\ \neg\circledcirc(A_1,\dots,A_n)\end{array}\ \begin{array}{c}\\ \mathfrak{D}_2^{\dagger}\\ A_i^{\dagger}\end{array}\ \begin{array}{c}[\neg A_i]^b\\ \mathfrak{D}_3^{\dagger}\\ B\end{array}\ \begin{array}{c}\\ \mathfrak{D}_4^{\S}\\ A_j^{\S}\end{array}\ \begin{array}{c}\\ \mathfrak{D}_5^{\S}\\ \neg A_j^{\S}\end{array}\ \begin{array}{c}[A_k^{\ddagger}]^c\\ \mathfrak{D}_6^{\ddagger}\\ B\end{array}\ \begin{array}{c}[\neg A_k^{\ddagger}]^d\\ \mathfrak{D}_7^{\ddagger}\\ B\end{array}\ \begin{array}{c}[A_l^{*}]^e\\ \mathfrak{D}_8^{*}\\ B\end{array}\ \begin{array}{c}\\ \mathfrak{D}_9^{*}\\ \neg A_l^{*}\end{array}}{B}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$$R_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle 0\big)^{b,c,d,e}\ \frac{\begin{array}{c}\\ \mathfrak{D}_1\\ \circledcirc(A_1,\dots,A_n)\end{array}\ \begin{array}{c}\\ \mathfrak{D}_2^{\dagger}\\ A_i^{\dagger}\end{array}\ \begin{array}{c}[\neg A_i]^b\\ \mathfrak{D}_3^{\dagger}\\ B\end{array}\ \begin{array}{c}\\ \mathfrak{D}_4^{\S}\\ A_j^{\S}\end{array}\ \begin{array}{c}\\ \mathfrak{D}_5^{\S}\\ \neg A_j^{\S}\end{array}\ \begin{array}{c}[A_k^{\ddagger}]^c\\ \mathfrak{D}_6^{\ddagger}\\ B\end{array}\ \begin{array}{c}[\neg A_k^{\ddagger}]^d\\ \mathfrak{D}_7^{\ddagger}\\ B\end{array}\ \begin{array}{c}[A_l^{*}]^e\\ \mathfrak{D}_8^{*}\\ B\end{array}\ \begin{array}{c}\\ \mathfrak{D}_9^{*}\\ \neg A_l^{*}\end{array}}{B}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

$$R^{\neg}_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle 0\big)^{a,b,c,d,e}\ \frac{\begin{array}{c}[\neg\circledcirc(A_1,\dots,A_n)]^a\\ \mathfrak{D}_1\\ B\end{array}\ \begin{array}{c}\\ \mathfrak{D}_2^{\dagger}\\ A_i^{\dagger}\end{array}\ \begin{array}{c}[\neg A_i]^b\\ \mathfrak{D}_3^{\dagger}\\ B\end{array}\ \begin{array}{c}\\ \mathfrak{D}_4^{\S}\\ A_j^{\S}\end{array}\ \begin{array}{c}\\ \mathfrak{D}_5^{\S}\\ \neg A_j^{\S}\end{array}\ \begin{array}{c}[A_k^{\ddagger}]^c\\ \mathfrak{D}_6^{\ddagger}\\ B\end{array}\ \begin{array}{c}[\neg A_k^{\ddagger}]^d\\ \mathfrak{D}_7^{\ddagger}\\ B\end{array}\ \begin{array}{c}[A_l^{*}]^e\\ \mathfrak{D}_8^{*}\\ B\end{array}\ \begin{array}{c}\\ \mathfrak{D}_9^{*}\\ \neg A_l^{*}\end{array}}{B}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$.

Notation 136. In order to save space let us write $\circledcirc(\vec{A})$ for $\circledcirc(A_1,\dots,A_n)$ (for any formulas $A_1,\dots,A_n$) and $f_{\circledcirc}(\vec{x})$ for $f_{\circledcirc}(x_1,\dots,x_n)$ (for any truth values $x_1,\dots,x_n$).

Theorem 137. *Let* $\mathbf{L}$ *be* $\mathbf{FDE}^{i(\circledcirc)_m}_{\neg}$. *Let* $x \in \{1,b,n,0\}$, $1 \leqslant i \leqslant 16$. *Then*:

- $f_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle\big) = x$ *iff both* $R_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle x\big)$ *and* $R^{\neg}_{\circledcirc}\big(\langle\mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle x\big)$ *are sound in* $\mathbf{L}$.

Proof. Similarly to Theorem 76. □

Theorem 138. *Let* $\mathbf{L}$ *be* $\mathbf{FDE}^{i(\circledcirc)_m}_{\neg}$, *for any* $1 \leqslant i \leqslant 16$. *For any set of formulas* Γ *and any formula* A, $\Gamma \vdash_{\mathbf{L}} A$ *in the natural deduction formulation of* $\mathbf{L}$ *iff* $\vdash_{\mathbf{L}} \Gamma \Rightarrow A$ *in the sequent formulation of* $\mathbf{L}$.

Proof. From 'left to right': by induction on the derivation in the natural deduction formulation of $\mathbf{L}$. From 'right to left': by induction on the derivation in the sequent formulation of $\mathbf{L}$. □

Corollary 139 (Adequacy)**.** *Let* $\mathbf{L}$ *be* $\mathbf{FDE}^{i(\circledcirc)_m}_{\neg}$, *for any* $1 \leqslant i \leqslant 16$. *Let* $\Gamma \subseteq \mathscr{F}^{\neg}_{(\circledcirc)_m}$ *and* $A \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$. *Then* $\Gamma \models_{\mathbf{L}} A$ *iff* $\Gamma \vdash_{\mathbf{L}} A$ *in the natural deduction formulation of* $\mathbf{L}$.

Proof. Follows from Theorems 129 and 138.

Another way to prove the Theorem is to give an argument analogous to the one used to establish Theorem 94: one will need to use Theorem 137 and define the notion of a $\mathbf{FDE}^{i(\circledcirc)_m}_{\neg}$-theorem. As an example, we define it for $\mathbf{FDE}^{1(\circledcirc)_m}_{\neg}$.

Let $\Gamma \subseteq \mathscr{F}^{\neg}_{(\circledcirc)_m}$ and $A, B \in \mathscr{F}^{\neg}_{(\circledcirc)_m}$. Then Γ is said to be an $\mathbf{FDE}^{1(\circledcirc)_m}_{\neg}$-*theory* iff the following conditions are fulfilled:

- (Γ_{N}) $\Gamma \neq \mathscr{F}^{\neg}_{(\circledcirc)_m}$ (non-triviality);
- (Γ_{Cl}) $\Gamma \vdash A$ implies $A \in \Gamma$ (closure under $\vdash$);
- $(\Gamma^{0}_{\mathrm{FDE}})$ if $f_{\circledcirc}(\langle\mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle) = 0$, then for each for each $i \in \mathfrak{t}$, for each $j \in \mathfrak{b}$, and for each $l \in \mathfrak{f}$,
  - $A_i \in \Gamma$, $A_j, \neg A_j \in \Gamma$, $\neg A_l \in \Gamma$ implies $\neg\circledcirc(A_1,\dots,A_n) \in \Gamma$, or $\neg A_i \in \Gamma$, or for some $k \in \mathfrak{n}$, $A_k \in \Gamma$ or $\neg A_k \in \Gamma$, or $A_l \in \Gamma$,
  - $\circledcirc(A_1,\dots,A_n) \in \Gamma$, $A_i \in \Gamma$, $A_j, \neg A_j \in \Gamma$, $\neg A_l \in \Gamma$ implies $\neg A_i \in \Gamma$, or for some $k \in \mathfrak{n}$, $A_k \in \Gamma$ or $\neg A_k \in \Gamma$, or $A_l \in \Gamma$;
- $(\Gamma^{n}_{\mathrm{FDE}})$ if $f_{\circledcirc}(\langle\mathfrak{t},\mathfrak{b},\mathfrak{n},\mathfrak{f}\rangle) = n$, then for each for each $i \in \mathfrak{t}$, for each $j \in \mathfrak{b}$, and for each $l \in \mathfrak{f}$,

 - $\neg\circledcirc(A_1, \dots, A_n) \in \Gamma$, $A_i \in \Gamma$, $A_j, \neg A_j \in \Gamma$, $\neg A_l \in \Gamma$ implies $\neg A_i \in \Gamma$, or for some $k \in \mathfrak{n}$, $A_k \in \Gamma$ or $\neg A_k \in \Gamma$, or $A_l \in \Gamma$,
 - $\circledcirc(A_1, \dots, A_n) \in \Gamma$, $A_i \in \Gamma$, $A_j, \neg A_j \in \Gamma$, $\neg A_l \in \Gamma$ implies $\neg A_i \in \Gamma$, or for some $k \in \mathfrak{n}$, $A_k \in \Gamma$ or $\neg A_k \in \Gamma$, or $A_l \in \Gamma$;

- (Γ^b_{FDE}) if $f_\circledcirc(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f} \rangle) = b$, then for each for each $i \in \mathfrak{t}$, for each $j \in \mathfrak{b}$, and for each $l \in \mathfrak{f}$,
  - $A_i \in \Gamma$, $A_j, \neg A_j \in \Gamma$, $\neg A_l \in \Gamma$ implies $\neg\circledcirc(A_1, \dots, A_n) \in \Gamma$, or $\neg A_i \in \Gamma$, or for some $k \in \mathfrak{n}$,$A_k \in \Gamma$ or $\neg A_k \in \Gamma$, or $A_l \in \Gamma$,
  - $A_i \in \Gamma$, $A_j, \neg A_j \in \Gamma$, $\neg A_l \in \Gamma$ implies $\circledcirc(A_1, \dots, A_n) \in \Gamma$, or $\neg A_i \in \Gamma$, or for some $k \in \mathfrak{n}$, $A_k \in \Gamma$ or $\neg A_k \in \Gamma$, or $A_l \in \Gamma$;

- (Γ^1_{FDE}) if $f_\circledcirc(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f} \rangle) = 1$, then for each for each $i \in \mathfrak{t}$, for each $j \in \mathfrak{b}$, and for each $l \in \mathfrak{f}$,
  - $A_i \in \Gamma$, $A_j, \neg A_j \in \Gamma$, $\neg A_l \in \Gamma$ implies $\circledcirc(A_1, \dots, A_n) \in \Gamma$ or $\neg A_i \in \Gamma$, or for some $k \in \mathfrak{n}$, $A_k \in \Gamma$ or $\neg A_k \in \Gamma$, or $A_l \in \Gamma$,
  - $\neg\circledcirc(A_1, \dots, A_n) \in \Gamma$, $A_i \in \Gamma$, $A_j, \neg A_j \in \Gamma$, $\neg A_l \in \Gamma$ implies $\neg A_i \in \Gamma$, or for some $k \in \mathfrak{n}$, $A_k \in \Gamma$ or $\neg A_k \in \Gamma$, or $A_l \in \Gamma$.

An *interpretation function* of A in Γ should be defined as follows:

$$e(A, \Gamma) = \begin{cases} 1 & \text{iff} \quad A \in \Gamma, \neg A \notin \Gamma; \\ b & \text{iff} \quad A \in \Gamma, \neg A \in \Gamma; \\ n & \text{iff} \quad A \notin \Gamma, \neg A \notin \Gamma; \\ 0 & \text{iff} \quad A \notin \Gamma, \neg A \in \Gamma. \end{cases}$$

The formulations and proofs of the required lemmas are similar to the formulations and proofs of Lemmas 89, 91, and 92. $\square$

The proof of normalisation for four-valued logics is analogous to the proof for three-valued logics. The notions of maximal formulas, degree of a formula, segment, length and degree of a segment, maximal formula, normal form, and rank of a deduction are understood according to Definitions 95, 96, 51, 52, 53, and 55, respectively.

As an example, we prove normalisation for $\mathbf{FDE}^{1(\circledcirc)_m}_{\neg} = \mathbf{FDE}^{(\circledcirc)_m}_{\neg}$.

In the case of the negation fragment of **FDE**, we have one case with the maximal formula: the maximal formula has the form $\neg\neg A$; the rules $(\neg\neg I)$ and $(\neg\neg E)$ are applied. The same case appears in **LP** and the reduction procedure is the same. In the case of the negation fragment of **FDE**, there is also one case with the maximal segment produced by the rules $(\neg\neg E)$ and $(\neg\neg I)$. The same case appears in **LP** and the permutation procedure is the same. Simplicity conversions and maximal segments described in the second clause of Definition 51 are considered similarly to the case of **LP**.

In the case of the negation fragment of **FDE** extended by n-ary operators $\circledcirc_1, \dots, \circledcirc_m$, we have the following cases.

Maximal formulas. Case 1. *Subcase 1.1.* The maximal formula $\circledcirc(\vec{A})$ produced by applications of the rules $R_\circledcirc\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f} \rangle 0\big)$ and $R_\circledcirc\big(\langle \mathfrak{t}', \mathfrak{b}', \mathfrak{n}', \mathfrak{f}' \rangle b\big)$ which we denote by $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively.

$$\mathfrak{R}_2^{a,f,g,h,o}\;\dfrac{\begin{array}{c} \mathfrak{R}_1^{b,c,d,e}\;\dfrac{[\circledcirc(\vec{A})]^a \quad \begin{array}{c} \mathfrak{D}_2^\dagger \\ A_i^\dagger \end{array} \quad \begin{array}{c} [\neg A_i]^b \\ \mathfrak{D}_3^\dagger \\ B \end{array} \quad \begin{array}{c} \mathfrak{D}_4^\S \\ A_j^\S \end{array} \quad \begin{array}{c} \mathfrak{D}_5^\S \\ \neg A_j^\S \end{array} \quad \begin{array}{c} [A_k^\ddagger]^c \\ \mathfrak{D}_6^\ddagger \\ B \end{array} \quad \begin{array}{c} [\neg A_k^\ddagger]^d \\ \mathfrak{D}_7^\ddagger \\ B \end{array} \quad \begin{array}{c} [A_l^*]^e \\ \mathfrak{D}_8^* \\ B \end{array} \quad \begin{array}{c} \mathfrak{D}_9^* \\ \neg A_l^* \end{array}}{B} \\ \mathfrak{H} \\ C \end{array} \qquad \mathfrak{X}}{C}$$

where $\mathfrak{X}$ stands for

$$
\begin{array}{cccccccc}
 & [\neg A_{i'}]^f & & & [A^{\flat}_{k'}]^g & [\neg A^{\flat}_{k'}]^h & [A^{\star}_{l'}]^o & \\
\mathfrak{E}^{\sharp}_2 & \mathfrak{E}^{\sharp}_3 & \mathfrak{E}^{\natural}_4 & \mathfrak{E}^{\natural}_5 & \mathfrak{E}^{\flat}_6 & \mathfrak{E}^{\flat}_7 & \mathfrak{E}^{\star}_8 & \mathfrak{E}^{\star}_9 \\
A^{\sharp}_{i'} & C & A^{\natural}_{j'} & \neg A^{\natural}_{j'} & C & C & C & \neg A^{\star}_{l'}
\end{array}
$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$; $^{\sharp}$ for each $i' \in \mathfrak{t}'$, $^{\natural}$ for each $j' \in \mathfrak{b}'$, $^{\flat}$ for each $k' \in \mathfrak{n}'$, * for each $l' \in \mathfrak{f}'$.

Recall that all the formulas

$$A_i, \neg A_i, A_j, \neg A_j, A_k, \neg A_k, A_l, \neg A_l, A_{i'}, \neg A_{i'}, A_{j'}, \neg A_{j'}, A_{k'}, \neg A_{k'}, A_{l'}, \neg A_{l'}$$

are either subformulas or negations of subformulas of $\circledcirc(\vec{A}) = \circledcirc(A_1, \ldots, A_n)$. We need to have more details about the other rules which are present in the system. Let us observe that this case can be reformulated in the following way.

$$
\mathfrak{R}_2^{a,f,g,h,o}\,\dfrac{\begin{array}{c}\mathfrak{R}_1^{b,c,d,e}\,\dfrac{[\circledcirc(\vec{A})]^a \quad X_1 \quad \ldots \quad X_n}{B} \\ \mathfrak{H} \\ C\end{array} \qquad Y_1 \quad \ldots \quad Y_n}{C}
$$

where $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are abbreviations for the following derivations, for any $t, u \in \{1, \ldots, n\}$.

$$
X_t = \begin{cases}
\begin{array}{cc} & [\neg A_t]^b \\ \mathfrak{D}_2 & \mathfrak{D}_3 \\ A_t & B \end{array} & \textit{iff} \quad t \in \mathfrak{t}; \\[2ex]
\begin{array}{cc} \mathfrak{D}_4 & \mathfrak{D}_5 \\ A_t & \neg A_t \end{array} & \textit{iff} \quad t \in \mathfrak{b}; \\[2ex]
\begin{array}{cc} [A_t]^c & [\neg A_t]^d \\ \mathfrak{D}_6 & \mathfrak{D}_7 \\ B & B \end{array} & \textit{iff} \quad t \in \mathfrak{n}; \\[2ex]
\begin{array}{cc} [A_t]^e & \\ \mathfrak{D}_8 & \mathfrak{D}_9 \\ B & \neg A_t \end{array} & \textit{iff} \quad t \in \mathfrak{f};
\end{cases}
\qquad
Y_u = \begin{cases}
\begin{array}{cc} & [\neg A_u]^f \\ \mathfrak{E}_2 & \mathfrak{E}_3 \\ A_u & C \end{array} & \textit{iff} \quad u \in \mathfrak{t}'; \\[2ex]
\begin{array}{cc} \mathfrak{E}_4 & \mathfrak{E}_5 \\ A_u & \neg A_u \end{array} & \textit{iff} \quad u \in \mathfrak{b}'; \\[2ex]
\begin{array}{cc} [A_u]^g & [\neg A_u]^h \\ \mathfrak{E}_6 & \mathfrak{E}_7 \\ C & C \end{array} & \textit{iff} \quad u \in \mathfrak{n}'; \\[2ex]
\begin{array}{cc} [A_u]^o & \\ \mathfrak{E}_8 & \mathfrak{E}_9 \\ C & \neg A_u \end{array} & \textit{iff} \quad u \in \mathfrak{f}'.
\end{cases}
$$

As follows from these equalities and soundness of our natural deduction systems, there is $s \in \{1, \ldots, n\}$ such that $X_s \neq Y_s$.

Then the following combinations are possible (we present them in the form of ordered pairs $\langle X_s, Y_s \rangle$):

$$\mathcal{C}_1 = \left\langle \begin{array}{cc} & [\neg A_s]^b \\ \mathfrak{D}_2 & \mathfrak{D}_3 \\ A_s & B \end{array} , \begin{array}{cc} \mathfrak{E}_4 & \mathfrak{E}_5 \\ A_s & \neg A_s \end{array} \right\rangle \qquad \mathcal{C}_2 = \left\langle \begin{array}{cc} & [\neg A_s]^b \\ \mathfrak{D}_2 & \mathfrak{D}_3 \\ A_s & B \end{array} , \begin{array}{cc} [A_s]^g & [\neg A_s]^h \\ \mathfrak{E}_6 & \mathfrak{E}_7 \\ C & C \end{array} \right\rangle$$

$$\mathcal{C}_3 = \left\langle \begin{array}{cc} & [\neg A_s]^b \\ \mathfrak{D}_2 & \mathfrak{D}_3 \\ A_s & B \end{array} , \begin{array}{cc} [A_s]^o & \\ \mathfrak{E}_8 & \mathfrak{E}_9 \\ C & \neg A_s \end{array} \right\rangle \qquad \mathcal{C}_4 = \left\langle \begin{array}{cc} \mathfrak{D}_4 & \mathfrak{D}_5 \\ A_s & \neg A_s \end{array} , \begin{array}{cc} & [\neg A_s]^f \\ \mathfrak{E}_2 & \mathfrak{E}_3 \\ A_s & C \end{array} \right\rangle$$

$$\mathcal{C}_5 = \left\langle \begin{array}{cc} \mathfrak{D}_4 & \mathfrak{D}_5 \\ A_s & \neg A_s \end{array} , \begin{array}{cc} [A_s]^g & [\neg A_s]^h \\ \mathfrak{E}_6 & \mathfrak{E}_7 \\ C & C \end{array} \right\rangle \qquad \mathcal{C}_6 = \left\langle \begin{array}{cc} \mathfrak{D}_4 & \mathfrak{D}_5 \\ A_s & \neg A_s \end{array} , \begin{array}{cc} [A_u]^o & \\ \mathfrak{E}_8 & \mathfrak{E}_9 \\ C & \neg A_s \end{array} \right\rangle$$

$$\mathcal{C}_7 = \left\langle \begin{array}{cc} [A_s]^c & [\neg A_s]^d \\ \mathfrak{D}_6 & \mathfrak{D}_7 \\ B & B \end{array} , \begin{array}{cc} & [\neg A_s]^f \\ \mathfrak{E}_2 & \mathfrak{E}_3 \\ A_s & C \end{array} \right\rangle \qquad \mathcal{C}_8 = \left\langle \begin{array}{cc} [A_s]^c & [\neg A_s]^d \\ \mathfrak{D}_6 & \mathfrak{D}_7 \\ B & B \end{array} , \begin{array}{cc} \mathfrak{E}_4 & \mathfrak{E}_5 \\ A_s & \neg A_s \end{array} \right\rangle$$

$$\mathcal{C}_9 = \left\langle \begin{array}{cc} [A_s]^c & [\neg A_s]^d \\ \mathfrak{D}_6 & \mathfrak{D}_7 \\ B & B \end{array} , \begin{array}{cc} [A_s]^o & \\ \mathfrak{E}_8 & \mathfrak{E}_9 \\ C & \neg A_s \end{array} \right\rangle \qquad \mathcal{C}_{10} = \left\langle \begin{array}{cc} [A_s]^o & \\ \mathfrak{D}_8 & \mathfrak{D}_9 \\ B & \neg A_s \end{array} , \begin{array}{cc} & [\neg A_s]^f \\ \mathfrak{E}_2 & \mathfrak{E}_3 \\ A_s & C \end{array} \right\rangle$$

$$\mathcal{C}_{11} = \left\langle \begin{array}{cc} [A_s]^o & \\ \mathfrak{D}_8 & \mathfrak{D}_9 \\ B & \neg A_s \end{array} , \begin{array}{cc} \mathfrak{E}_4 & \mathfrak{E}_5 \\ A_s & \neg A_s \end{array} \right\rangle \qquad \mathcal{C}_{12} = \left\langle \begin{array}{cc} [A_s]^o & \\ \mathfrak{D}_8 & \mathfrak{D}_9 \\ B & \neg A_s \end{array} , \begin{array}{cc} [A_s]^g & [\neg A_s]^h \\ \mathfrak{E}_6 & \mathfrak{E}_7 \\ C & C \end{array} \right\rangle$$

Let us consider the case $\mathcal{C}_1$.

$$\mathfrak{R}_1^{b,c,d,e} \quad \dfrac{\mathfrak{R}_2^{a,f,g,h,o}\ \begin{array}{c} \dfrac{[\circledcirc(\vec{A})]^a \quad X_1 \ldots X_{s-1} \quad \begin{array}{c} \mathfrak{D}_2 \\ A_s \end{array} \quad \begin{array}{c} [\neg A_s]^b \\ \mathfrak{D}_3 \\ B \end{array} \quad X_{s+1} \ldots X_n}{B} \\ \mathfrak{H} \\ C \end{array} \qquad Y_1 \ldots Y_{s-1} \quad \begin{array}{c} \mathfrak{E}_4 \\ A_s \end{array} \quad \begin{array}{c} \mathfrak{E}_5 \\ \neg A_s \end{array} \quad Y_{s+1} \ldots Y_n}{C}$$

Then we can introduce the following reduction procedure:

$$\mathfrak{R}_2^{a,f,g,h,o}\ \dfrac{\begin{array}{c} \mathfrak{E}_5 \\ \neg A_s \\ \mathfrak{D}_3 \\ B \\ \mathfrak{H} \\ C \end{array} \quad Y_1 \ldots Y_{s-1} \quad \begin{array}{c} \mathfrak{E}_4 \\ A_s \end{array} \quad \begin{array}{c} \mathfrak{E}_5 \\ \neg A_s \end{array} \quad Y_{s+1} \ldots Y_n}{C}$$

The other cases $\mathcal{C}_2, \ldots, \mathcal{C}_{12}$ are considered similarly.

The following subcases are considered similarly.

Subcase 1.2. The maximal formula $\circledcirc(\vec{A})$ is produced by applications of $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f} \rangle n\big)$ and $R_{\circledcirc}\big(\langle \mathfrak{t}', \mathfrak{b}', \mathfrak{n}', \mathfrak{f}' \rangle b\big)$.

Subcase 1.3. The maximal formula $\circledcirc(\vec{A})$ is produced by $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f} \rangle 0\big)$ and $R_{\circledcirc}\big(\langle \mathfrak{t}', \mathfrak{b}', \mathfrak{n}', \mathfrak{f}' \rangle 1\big)$.

Subcase 1.4. The maximal formula $\circledcirc(\vec{A})$ is produced by $R_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f} \rangle n\big)$ and $R_{\circledcirc}\big(\langle \mathfrak{t}', \mathfrak{b}', \mathfrak{n}', \mathfrak{f}' \rangle 1\big)$.

Subcase 1.5. The maximal formula $\neg\circledcirc(\vec{A})$ is produced by $R^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f} \rangle 1\big)$ and $R^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}', \mathfrak{b}', \mathfrak{n}', \mathfrak{f}' \rangle b\big)$.

Subcase 1.6. The maximal formula $\neg\circledcirc(\vec{A})$ is produced by $R^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f} \rangle n\big)$ and $R^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}', \mathfrak{b}', \mathfrak{n}', \mathfrak{f}' \rangle b\big)$.

Subcase 1.7. The maximal formula $\neg\circledcirc(\vec{A})$ is produced by $R^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f} \rangle 1\big)$ and $R^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}', \mathfrak{b}', \mathfrak{n}', \mathfrak{f}' \rangle 0\big)$.

Subcase 1.8. The maximal formula $\neg\circledcirc(\vec{A})$ is produced by $R^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}, \mathfrak{b}, \mathfrak{n}, \mathfrak{f} \rangle n\big)$ and $R^{\neg}_{\circledcirc}\big(\langle \mathfrak{t}', \mathfrak{b}', \mathfrak{n}', \mathfrak{f}' \rangle 0\big)$.

Maximal segments. Case 2. The maximal segment with the formula $\neg\neg B$ is produced by applications of any rule for $\circledcirc$ and $(\neg\neg E)$ denoted as $\mathfrak{R}_1$ and $\mathfrak{R}_2$, respectively.

$$\mathfrak{R}_2^f\ \dfrac{\mathfrak{R}_1^{a,b,c,d,e}\ \dfrac{\mathfrak{X}\quad \begin{matrix}\mathfrak{D}_2^\dagger\\ A_i^\dagger\end{matrix}\quad \begin{matrix}[\neg A_i^\dagger]^b\\ \mathfrak{D}_3^\dagger\\ \neg\neg B\end{matrix}\quad \begin{matrix}\mathfrak{D}_4^\S\\ A_j^\S\end{matrix}\quad \begin{matrix}\mathfrak{D}_5^\S\\ \neg A_j^\S\end{matrix}\quad \begin{matrix}[A_k^\ddagger]^c\\ \mathfrak{D}_6^\ddagger\\ \neg\neg B\end{matrix}\quad \begin{matrix}[\neg A_k^\ddagger]^d\\ \mathfrak{D}_7^\ddagger\\ \neg\neg B\end{matrix}\quad \begin{matrix}[A_l^*]^e\\ \mathfrak{D}_8^*\\ \neg\neg B\end{matrix}\quad \begin{matrix}\mathfrak{D}_9^*\\ \neg A_l^*\end{matrix}}{\neg\neg B}\quad \begin{matrix}[B]^f\\ \mathfrak{D}_1\\ C\end{matrix}}{C}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$, where $\mathfrak{X}$ stands for one of the following options depending on which rule for $\circledcirc$ we have:

$$\begin{matrix}[\circledcirc(\vec{A})]^a\\ \mathfrak{E}_1\\ \neg\neg B\end{matrix}\quad \begin{matrix}[\neg\circledcirc(\vec{A})]^a\\ \mathfrak{E}_2\\ \neg\neg B\end{matrix}\quad \begin{matrix}\mathfrak{E}_3\\ \circledcirc(\vec{A})\end{matrix}\quad \begin{matrix}\mathfrak{E}_4\\ \neg\circledcirc(\vec{A})\end{matrix}$$

The permutation procedure is changing the order of applications of the rules. Let us introduce the following abbreviations $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_4$:

$$\mathfrak{A}_1 = \mathfrak{R}_2^f\ \dfrac{\begin{matrix}[\neg A_i^\dagger]^b\\ \mathfrak{D}_3^\dagger\\ \neg\neg B\end{matrix}\quad \begin{matrix}[B]^f\\ \mathfrak{D}_1\\ C\end{matrix}}{C}\qquad \mathfrak{A}_2 = \mathfrak{R}_2^f\ \dfrac{\begin{matrix}[A_k^\ddagger]^c\\ \mathfrak{D}_6^\ddagger\\ \neg\neg B\end{matrix}\quad \begin{matrix}[B]^f\\ \mathfrak{D}_1\\ C\end{matrix}}{C}$$

$$\mathfrak{A}_3 = \mathfrak{R}_2^f\ \dfrac{\begin{matrix}[\neg A_k^\ddagger]^d\\ \mathfrak{D}_7^\ddagger\\ \neg\neg B\end{matrix}\quad \begin{matrix}[B]^f\\ \mathfrak{D}_1\\ C\end{matrix}}{C}\qquad \mathfrak{A}_4 = \mathfrak{R}_2^f\ \dfrac{\begin{matrix}[A_l^*]^e\\ \mathfrak{D}_8^*\\ \neg\neg B\end{matrix}\quad \begin{matrix}[B]^f\\ \mathfrak{D}_1\\ C\end{matrix}}{C}$$

Then we have

$$\mathfrak{R}_1^{a,b,c,d,e}\ \dfrac{\mathfrak{Y}\quad \begin{matrix}\mathfrak{D}_2^\dagger\\ A_i^\dagger\end{matrix}\quad \mathfrak{A}_1\quad \begin{matrix}\mathfrak{D}_4^\S\\ A_j^\S\end{matrix}\quad \begin{matrix}\mathfrak{D}_5^\S\\ \neg A_j^\S\end{matrix}\quad \mathfrak{A}_2\quad \mathfrak{A}_3\quad \mathfrak{A}_4\quad \begin{matrix}\mathfrak{D}_9^*\\ \neg A_l^*\end{matrix}}{C}$$

† for each $i \in \mathfrak{t}$, § for each $j \in \mathfrak{b}$, ‡ for each $k \in \mathfrak{n}$, * for each $l \in \mathfrak{f}$, where $\mathfrak{Y}$ stands for one of the following options depending on which rule for $\circledcirc$ we have:

$$\mathfrak{R}_2^f\ \dfrac{\begin{matrix}[\circledcirc(\vec{A})]^a\\ \mathfrak{E}_1\\ \neg\neg B\end{matrix}\quad \begin{matrix}[B]^f\\ \mathfrak{D}_1\\ C\end{matrix}}{C}\quad \mathfrak{R}_2^f\ \dfrac{\begin{matrix}[\neg\circledcirc(\vec{A})]^a\\ \mathfrak{E}_2\\ \neg\neg B\end{matrix}\quad \begin{matrix}[B]^f\\ \mathfrak{D}_1\\ C\end{matrix}}{C}\quad \begin{matrix}\mathfrak{E}_3\\ \circledcirc(\vec{A})\end{matrix}\quad \begin{matrix}\mathfrak{E}_4\\ \neg\circledcirc(\vec{A})\end{matrix}$$

Case 3. The maximal segment with the formula $\circledcirc_1(\vec{B})$ (or the formula $\neg\circledcirc_1(\vec{B})$) is produced by applications of a rule for $\circledcirc_1$ and a rule for $\circledcirc_2$ ($\circledcirc_1$ and $\circledcirc_2$ can be distinct operators, but can coincide). This case is considered similarly to the previous one: the permutation procedure is to change the order of applications of the rules.

As for the cases of the other fifteen negations, all the maximal segments that can be produced by the rules for these negations and the rules for n-ary operators are treated in an analogous way: the permutation procedure is to change the order of applications of the rules. The cases with maximal formulas are treated similarly to the cases of **FDE**, **LP**, and $\mathbf{K_3}$.

Theorem 140. *Let* $\mathbf{L} \in \{\mathbf{FDE}^{i}_{\neg}, \mathbf{FDE}^{i(\circledcirc)_m}_{\neg}\}$, $1 \leqslant i \leqslant 16$. *Any deduction in* $\mathbf{L}$ *can be converted into a deduction in normal form.*

Proof. Similarly to Theorem 100. □

Theorem 141. *Let* $\mathbf{L} \in \{\mathbf{FDE}^{i}_{\neg}, \mathbf{FDE}^{i(\circledcirc)_m}_{\neg}\}$, $1 \leqslant i \leqslant 16$. *Deductions in normal forms in* $\mathbf{L}$ *have the negation subformula property.*

Proof. Similarly to Theorem 104. □

Chapter 4

Proof systems for selected many-valued modal logics

4.1 Preliminaries and semantics

This section is devoted to the combination of many-valued logics with modal ones. The first attempt to provide such logics is an extension of Łukasiewicz's three-valued logic $\mathbf{Ł_3}$ by the following tabular modalities:

A	$\Box A$	$\Diamond A$
1	1	1
$1/2$	0	1
0	0	0

Later on, Łukasiewicz proposed a four-valued approach to modal logic with tabular modalities [116]. Both approaches are not popular in modern logic, because of some philosophically strange principles they have (e.g. $A \to (A \to \Box A)$ in the three-valued logic and $\Box A \to (\Box B \leftrightarrow \Diamond B)$ in the four-valued). However, one can find some modern papers about Łukasiewicz's modal many-valued logics: e.g., Font and Hájek [57] study algebraic properties of Łukasiewicz's four-valued modal logic and conclude that some of its strange properties are connected with Aristotelian modal syllogistic; Méndez and Robles show two ways how these strange properties could be eliminated: either by the replacement of the propositional part of Łukasiewicz's modal four-valued logic by Brady's [21] relevant four-valued logic $\mathbf{BN_4}$ [121] or by using a paraconsistent four-valued logic **PŁ4** [122].

Nowadays, many-valued modal logic usually uses many-valued Kripke semantics as a tool for defining modalities. Among the first papers regarding this approach are Fitting's ones [52, 53]. Modal extensions of **FDE** and related logics were also studied by Frankowski [58], Goble [66], Priest [164, 165], Odintsov and Wansing [136, 137] as well as Odintsov, Skurt, and Wansing [135], Odintsov and Latkin [132], Odintsov and Speranski [133, 134], Sedlár [172], Rivieccio, Jung, and Jansana [169].

We will use many-valued modalities with the following truth and falsity conditions (which are applicable for both three- and four-valued cases)[1]:

- $1 \in \vartheta(\Box A, x)$ iff $\forall_{y \in W}(R(x,y)$ implies $1 \in \vartheta(A,y))$,
- $0 \in \vartheta(\Box A, x)$ iff $\exists_{y \in W}(R(x,y)$ and $0 \in \vartheta(A,y))$,
- $1 \in \vartheta(\Diamond A, x)$ iff $\exists_{y \in W}(R(x,y)$ and $1 \in \vartheta(A,y))$,

[1] In defining the many-valued necessity operator in such a way we follow Odintsov and Wansing's approach [136]; other modalities are defined in a similar fashion.

- $0 \in \vartheta(\Diamond A, x)$ iff $\forall_{y \in W}(R(x, y)$ implies $0 \in \vartheta(A, y))$,
- $1 \in V(\rhd A, x)$ iff $\forall_{y \in W}(R(x, y)$ implies $1 \in V(A, y))$ or $\forall_{y \in W}(R(x, y)$ implies $1 \notin V(A, y))$,
- $0 \in V(\rhd A, x)$ iff $\exists_{y \in W}(R(x, y)$ and $0 \notin V(A, y))$ and $\exists_{y \in W}(R(x, y)$ and $0 \in V(A, y))$,
- $1 \in V(\blacktriangleright A, x)$ iff $\exists_{y \in W}(R(x, y)$ and $1 \notin V(A, y))$ and $\exists_{y \in W}(R(x, y)$ and $1 \in V(A, y))$,
- $0 \in V(\blacktriangleright A, x)$ iff $\forall_{y \in W}(R(x, y)$ implies $0 \notin V(A, y))$ or $\forall_{y \in W}(R(x, y)$ implies $0 \in V(A, y))$,
- $1 \in \vartheta(\circ A, x)$ iff $1 \notin \vartheta(A, x)$ or $\forall_{y \in W}(R(x, y)$ implies $1 \in \vartheta(A, y))$,
- $0 \in \vartheta(\circ A, x)$ iff $0 \notin \vartheta(A, x)$ and $\exists_{y \in W}(R(x, y)$ and $0 \in \vartheta(A, y))$,
- $1 \in \vartheta(\bullet A, x)$ iff $1 \in \vartheta(A, x)$ and $\exists_{y \in W}(R(x, y)$ and $1 \notin \vartheta(A, y))$,
- $0 \in \vartheta(\bullet A, x)$ iff $0 \in \vartheta(A, x)$ or $\forall_{y \in W}(R(x, y)$ implies $0 \notin \vartheta(A, y))$,
- $1 \in \vartheta(\tilde{\circ} A, x)$ iff $1 \in \vartheta(A, x)$ or $\forall_{y \in W}(R(x, y)$ implies $1 \notin \vartheta(A, y))$,
- $0 \in \vartheta(\tilde{\circ} A, x)$ iff $0 \in \vartheta(A, x)$ and $\exists_{y \in W}(R(x, y)$ and $0 \notin \vartheta(A, y))$,
- $1 \in \vartheta(\tilde{\bullet} A, x)$ iff $1 \notin \vartheta(A, x)$ and $\exists_{y \in W}(R(x, y)$ and $1 \in \vartheta(A, y))$,
- $0 \in \vartheta(\tilde{\bullet} A, x)$ iff $0 \notin \vartheta(A, x)$ or $\forall_{y \in W}(R(x, y)$ implies $0 \in \vartheta(A, y))$,
- $1 \in \vartheta(\sim A, x)$ iff $\exists_{y \in W}(R(x, y)$ and $1 \notin \vartheta(A, y))$,
- $0 \in \vartheta(\sim A, x)$ iff $\forall_{y \in W}(R(x, y)$ implies $0 \notin \vartheta(A, y))$,
- $1 \in \vartheta(\dot{\sim} A, x)$ iff $\forall_{y \in W}(R(x, y)$ implies $1 \notin \vartheta(A, y))$,
- $0 \in \vartheta(\dot{\sim} A, x)$ iff $\exists_{y \in W}(R(x, y)$ and $0 \notin \vartheta(A, y))$.

The remarkable feature of this approach is that we postulate both truth and falsity conditions. It is enough for the classical case to give only the truth part of the conditions, the falsity part is deducible. However, since we have more values now, the truth and falsity conditions have become independent from each other. This feature has its representation on the syntactical level as well: we have to give rules not only for the connectives themselves, but for the negations of the connectives. Again, in the classical case, such rules for the negations of connectives are derivable, but in the many-valued case, they are independent.

4.2 Hypersequent calculi for modal many-valued logics

Hypersequent calculi for modal many-valued logics are obtained from the hypersequent calculi for two-valued modal logics by replacing the classical rules with the rules (in their hypersequent formulations) for three- and four-valued logics considered in the previous chapter and adding the rules for the negated modalities, which are given below.

For the negated necessity and possibility operators, we use Kamide's ones [90] which are an adaptation of the rules of $\mathbf{MML}_n^{\mathbf{S5}}$ from [71], as he notes [90, p. 35]:

$$(\neg\Box \Rightarrow)\ \frac{\neg A \Rightarrow \mid H}{\neg\Box A \Rightarrow \mid H} \qquad (\Rightarrow \neg\Box)\ \frac{\Gamma \Rightarrow \Delta, \neg A \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \neg\Box A \mid H}$$

$$(\neg\Diamond \Rightarrow)\ \frac{\neg A, \Gamma \Rightarrow \Delta \mid H}{\neg\Diamond A \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg\Diamond)\ \frac{\Rightarrow \neg A \mid H}{\Rightarrow \neg\Diamond A \mid H}$$

As for non-standard modalities, we offer the following rules for their negations:

$$(\neg\triangleright\Rightarrow)\ \frac{\Rightarrow\neg A\mid\neg A\Rightarrow\ \mid H}{\neg\triangleright A\Rightarrow\ \mid H}\qquad(\Rightarrow\neg\triangleright)\ \frac{\neg A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,\neg A\mid G}{\Rightarrow\neg\triangleright A\mid\Gamma\Rightarrow\Delta\mid\Theta\Rightarrow\Lambda\mid H\mid G}$$

$$(\neg\blacktriangleright\Rightarrow)\ \frac{\neg A,\Gamma\Rightarrow\Delta\mid H\qquad\Theta\Rightarrow\Lambda,\neg A\mid G}{\neg\blacktriangleright A\Rightarrow\ \mid\Gamma\Rightarrow\Delta\mid\Theta\Rightarrow\Lambda\mid H\mid G}\qquad(\Rightarrow\neg\blacktriangleright)\ \frac{\Rightarrow\neg A\mid\neg A\Rightarrow\ \mid H}{\Rightarrow\ \neg\blacktriangleright A\mid H}$$

$$(\neg\circ\Rightarrow)\ \frac{\neg A\Rightarrow\ \mid\Gamma\Rightarrow\Delta,\neg A\mid H}{\neg\circ A,\Gamma\Rightarrow\Delta\mid H}\qquad(\Rightarrow\neg\circ)\ \frac{\Gamma\Rightarrow\Delta,\neg A\mid H\qquad\neg A,\Theta\Rightarrow\Lambda\mid G}{\Theta\Rightarrow\Lambda,\neg\circ A\mid\Gamma\Rightarrow\Delta\mid H\mid G}$$

$$(\neg\bullet\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,\neg A\mid H\qquad\neg A,\Theta\Rightarrow\Lambda\mid G}{\neg\bullet A,\Theta\Rightarrow\Lambda\mid\Gamma\Rightarrow\Delta\mid H\mid G}\ (\Rightarrow\neg\bullet)\ \frac{\neg A\Rightarrow\ \mid\Gamma\Rightarrow\Delta,\neg A\mid H}{\Gamma\Rightarrow\Delta,\neg\bullet A\mid H}$$

$$(\neg\widetilde{\circ}\Rightarrow)\ \frac{\neg A,\Gamma\Rightarrow\Delta\mid\ \Rightarrow\neg A\mid H}{\neg\widetilde{\circ}A,\Gamma\Rightarrow\Delta\mid H}\qquad(\Rightarrow\neg\widetilde{\circ})\ \frac{\Gamma\Rightarrow\Delta,\neg A\mid H\qquad\neg A,\Theta\Rightarrow\Lambda\mid G}{\Gamma\Rightarrow\Delta,\neg\widetilde{\circ}A\mid\Theta\Rightarrow\Lambda\mid H\mid G}$$

$$(\neg\widetilde{\bullet}\Rightarrow)\ \frac{\Gamma\Rightarrow\Delta,\neg\widetilde{\bullet}A\mid H\qquad\neg\widetilde{\bullet}A,\Theta\Rightarrow\Lambda\mid G}{\neg\widetilde{\bullet}A,\Gamma\Rightarrow\Delta\mid\Theta\Rightarrow\Lambda\mid H\mid G}\qquad(\Rightarrow\neg\widetilde{\bullet})\ \frac{\neg A,\Gamma\Rightarrow\Delta\mid\ \Rightarrow\neg A\mid H}{\Gamma\Rightarrow\Delta,\neg\widetilde{\bullet}A\mid H}$$

$$(\neg\sim\Rightarrow)\ \frac{\neg A,\Gamma\Rightarrow\Delta\mid H}{\neg\sim A\Rightarrow\ \mid\Gamma\Rightarrow\Delta\mid H}\qquad(\Rightarrow\neg\sim)\ \frac{\Rightarrow\neg A\mid H}{\Rightarrow\neg\sim A\mid H}$$

$$(\neg\dot{\sim}\Rightarrow)\ \frac{\Rightarrow\neg A\mid H}{\neg\dot{\sim}A\Rightarrow\ \mid H}\qquad(\Rightarrow\neg\dot{\sim})\ \frac{\neg A,\Gamma\Rightarrow\Delta\mid H}{\Rightarrow\neg\dot{\sim}A\mid\Gamma\Rightarrow\Delta\mid H}$$

Let $\clubsuit\in\{\Box,\Diamond,\triangleright,\blacktriangleright,\circ,\bullet,\widetilde{\circ},\widetilde{\bullet},\sim,\dot{\sim}\}$. Let $\mathbf{X}\in\{\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)m},\mathbf{LP}_{\neg}^{i(\circledcirc)m},\mathbf{FDE}_{\neg}^{j(\circledcirc)m}\}$, where $1\leqslant i\leqslant 4$, $1\leqslant j\leqslant 16$. By $\mathbf{XS5}^{\clubsuit}$ we denote a logic which is the logic $\mathbf{X}$ supplied with $\mathbf{S5}$-style many-valued modalities, defined according to the definitions listed in Section 4.1. Similar notation will be used for the logics based on $\mathbf{K}$-, $\mathbf{S4}$-, and other types of modalities.

Now we are going to present a Hintikka-style completeness proof for the modal many-valued logics in questions. Notice that in order to obtain the negation subformula property we use the formulations of these logics with the axioms $\Rightarrow A,\neg A$ ($\mathbf{LP}$-style logics) and $A,\neg A\Rightarrow$ ($\mathbf{K_3}$-style logics).

Theorem 142 (Strong soundness)**.** *Let* $\mathbf{X}\in\{\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)m},\mathbf{LP}_{\neg}^{i(\circledcirc)m},\mathbf{FDE}_{\neg}^{j(\circledcirc)m}\}$*, where* $1\leqslant i\leqslant 4$*,* $1\leqslant j\leqslant 16$*. Let* $\clubsuit\in\{\Box,\Diamond,\triangleright,\blacktriangleright,\circ,\bullet,\widetilde{\circ},\widetilde{\bullet},\sim,\dot{\sim}\}$ *and* $\mathbf{L}=\mathbf{XS5}^{\clubsuit}$*. For each finite set of hypersequents* $\mathscr{H}\cup\{H\}$*, if* $\mathscr{H}\vdash_{\mathbb{HSL}}H$*, then* $\mathscr{H}\models_{\mathbf{L}}H$*.*

Proof. The propositional case follows from Theorems 119 and 128. The modal case can be proven similarly to Theorem 5. □

Theorem 143 (Strong completeness)**.** *Let* $\mathbf{L}\in\{\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)m},\mathbf{LP}_{\neg}^{i(\circledcirc)m},\mathbf{FDE}_{\neg}^{j(\circledcirc)m}\}$*, where* $1\leqslant i\leqslant 4$*,* $1\leqslant j\leqslant 16$*. Let* $\clubsuit\in\{\Box,\Diamond,\triangleright,\blacktriangleright,\circ,\bullet,\widetilde{\circ},\widetilde{\bullet},\sim,\dot{\sim}\}$ *and* $\mathbf{L}=\mathbf{S5}^{\clubsuit}$*. For each finite set of hypersequents* $\mathscr{H}\cup\{H\}$*, if* $\mathscr{H}\models_{\mathbf{L}}H$*, then* $\mathscr{H}\vdash^{\mathrm{cf}}_{\mathbb{HSL}}H$*.*

Proof. Similarly to Theorem 6. We again adapt Avron and Lahav's [9] completeness proof for $\mathbf{Z}$ and use all the constructions and notation from the proof of Theorem 6 with some minor changes regarding the definition of a valuation which we explain below.

Suppose that $\mathscr{H}\nvdash^{\mathrm{cf}}_{\mathbb{HSL}}H$. We construct a model of $\mathscr{H}$ which is not a model of H.

For each $p\in\mathcal{P}$, let us define a valuation ϑ in the subsequent way (for the case $\mathbf{LP}_{\neg}^{i(\circledcirc)m}$):

- $v(p)={}^1\!/_2$ iff $p\in\Gamma_w$ and $\neg p\in\Gamma_w$,
- $v(p)=1$ iff $p\in\Gamma_w$ and $\neg p\notin\Gamma_w$,

- $v(p) = 0$ iff $p \notin \Gamma_w$ and $\neg p \in \Gamma_w$.

The valuation ϑ is well-defined, since $\Rightarrow p, \neg p$ is provable in the sequent calculus for $\mathbf{LP}_{\neg}^{i(\circledcirc)_m}$. For the case $\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}$, the condition for $1/2$ should be changed:

- $v(p) = 1/2$ iff $p \notin \Gamma_w$ and $\neg p \notin \Gamma_w$.

The valuation ϑ is well-defined, since $p, \neg p \Rightarrow$ is provable in the sequent calculus for $\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}$. In the case of $\mathbf{FDE}_{\neg}^{j(\circledcirc)_m}$, the condition for $1/2$ should be replaced with the following ones:

- $v(p) = b$ iff $p \in \Gamma_w$ and $\neg p \in \Gamma_w$,

- $v(p) = n$ iff $p \notin \Gamma_w$ and $\neg p \notin \Gamma_w$.

We need to prove the following implications, for every formula A:

- if $A \in \Delta_w$, then $1 \notin \vartheta(A, w)$;
- if $A \in \Gamma_w$, then $1 \in \vartheta(A, w)$;
- if $\neg A \in \Delta_w$, then $0 \notin \vartheta(A, w)$;
- if $\neg A \in \Gamma_w$, then $0 \in \vartheta(A, w)$.

As an example, we provide the proof for $\mathbf{LP}_{\neg}^{1(\circledcirc)_m} = \mathbf{LP}_{\neg}^{(\circledcirc)_m}$, $\mathbf{LP}_{\neg}^{3(\circledcirc)_m} = \mathbf{DG}_{\mathbf{3}\neg}^{(\circledcirc)_m}$, $\mathbf{LP}_{\neg}^{4(\circledcirc)_m} = \mathbf{DP}_{\mathbf{3}\neg}^{(\circledcirc)_m}$, and the cases of some modalities which can be added to any of the logics in question (these cases seem to be the most representative ones).

The proof is by induction on the complexity of A. The basic case (i.e., $A \in \mathcal{P}$) follows from the definition of ϑ.

Let A be $\neg B$. Suppose that $A \in \Delta_w$, i.e., $\neg B \in \Delta_w$. By the induction hypothesis on B, we have $0 \notin \vartheta(B, w)$. Thus, $1 \notin \vartheta(\neg B, w)$.

Suppose that $A \in \Gamma_w$, i.e., $\neg B \in \Gamma_w$. By the induction hypothesis on B, $0 \in \vartheta(B, w)$. Thus, $1 \in \vartheta(\neg B, w)$.

Suppose that $\neg A \in \Delta_w$, i.e., $\neg\neg B \in \Delta_w$. In the case of $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$, suppose that $B \notin \Delta_w$. Since w is maximal, $\Gamma_w \Rightarrow \Delta_w, B \notin H^*$. Since H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, B$. By the rule $(\Rightarrow \neg\neg)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \neg\neg B$. Since $\neg\neg B \in \Delta_w$, by (IC$\Rightarrow$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w$, i.e., $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid w$. Since $w \in H^*$, by (EC), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Thus, $B \in \Delta_w$. By the induction hypothesis, $1 \notin \vartheta(B, w)$. Hence, $0 \notin \vartheta(\neg B, w)$.

In the case of the logic $\mathbf{DG}_{\mathbf{3}\neg}^{(\circledcirc)_m}$, suppose that $\neg B \notin \Gamma_w$. Since w is maximal and H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg B, \Gamma_w \Rightarrow \Delta_w$. Since $\neg\neg B \in \Delta_w$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg\neg B, \Gamma_w \Rightarrow \Delta_w$. Then, by the rule (EM$\Rightarrow$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w$. Contradiction. Thus, $\neg B \in \Gamma_w$. By the induction hypothesis on B, $0 \in v(B, w)$. Thus, $0 \notin v(\neg B, w)$.

In the case of the logic $\mathbf{DP}_{\mathbf{3}\neg}^{(\circledcirc)_m}$, suppose that $B \notin \Gamma_w$ or $\neg B \notin \Gamma_w$. Since w is maximal and H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B, \Gamma_w \Rightarrow \Delta_w$ or $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg B, \Gamma_w \Rightarrow \Delta_w$. In the former case, we apply the rule $(\Rightarrow \neg_4\neg_4)$ and get $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \neg\neg B$. As in the case of $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$, it entails a contradiction. in the latter case, notice that since $\neg\neg B \in \Delta_w$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg\neg B, \Gamma_w \Rightarrow \Delta_w$, then apply (EM$\Rightarrow$) and get $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w$. Contradiction. Thus, $B \in \Gamma_w$ and $\neg B \in \Gamma_w$. By the induction hypothesis on B, $1 \in v(B, w)$ and $0 \in v(B, w)$, i.e., $v(B, w) = 1/2$. Thus, $v(\neg B, w) = 1$, hence $0 \notin v(\neg B, w)$.

Suppose that $\neg A \in \Gamma_w$, i.e., $\neg\neg B \in \Gamma_w$. For the case of the logic $\mathbf{LP}_{\neg}^{(\circledcirc)_m}$ suppose that $B \notin \Gamma_w$. Since w is maximal, $B, \Gamma_w \Rightarrow \Delta_w \notin H^*$. Since H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B, \Gamma_w \Rightarrow \Delta_w$. By the rule $(\neg\neg \Rightarrow)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg\neg B, \Gamma_w \Rightarrow \Delta_w$. Since $\neg\neg B \in \Gamma_w$, by (IC$\Rightarrow$), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid w$. By (EC), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Thus, $B \in \Gamma_w$. By the induction hypothesis, $1 \in v(B, w)$. Hence, $0 \in v(\neg B, w)$.

For the case of the logic $\mathbf{DG}_{\mathbf{3}_\neg}^{(\circledcirc)_m}$ suppose that $\neg B \notin \Delta_w$. Since w is maximal and H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \neg B$. By the rule $(\neg\neg \Rightarrow_{\mathbf{G}})$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg\neg B, \Gamma_w \Rightarrow \Delta_w$. As in the case of $\mathbf{LP}_\neg^{(\circledcirc)_m}$, it entails a contradiction. Thus, $\neg B \in \Delta_w$. By the induction hypothesis on B, $\mathsf{0} \notin v(B, w)$. Hence, $\mathsf{0} \in v(\neg B, w)$.

For the case of the logic $\mathbf{DP}_{\mathbf{3}_\neg}^{(\circledcirc)_m}$ suppose that $B \notin \Delta_w$ and $\neg B \notin \Delta_w$. Since w is maximal and H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, B$ and $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \neg B$. By the rule $(\neg_4\neg_4 \Rightarrow)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg\neg B, \Gamma_w \Rightarrow \Delta_w$. As in the case of $\mathbf{LP}_\neg^{(\circledcirc)_m}$, it entails a contradiction. Thus, $B \in \Delta_w$ or $\neg B \in \Delta_w$. By the induction hypothesis on B, $\mathsf{1} \notin v(B, w)$ or $\mathsf{0} \notin v(B, w)$, i.e., $v(B, w) \neq 1/2$. Thus, $v(\neg B, w) \neq 1$, and hence $\mathsf{0} \in v(\neg B, w)$.

Let A be $\circledcirc(B_1, \ldots, B_n)$. Suppose that $A \in \Delta_w$, i.e., $\circledcirc(B_1, \ldots, B_n) \in \Delta_w$. Let for each l, such that $1 \leqslant l \leqslant n$, $v(B_l) = x_l$, and $f_\circledcirc(x_1, \ldots, x_n) = y$. Suppose that $\mathsf{1} \in v(\circledcirc(B_1, \ldots, B_n), w)$. Thus, $\mathsf{1} \in f_\circledcirc(v(B_1), \ldots, v(B_n))$, $\mathsf{1} \in f_\circledcirc(x_1, \ldots, x_n)$, and $\mathsf{1} \in y$. Suppose that $\neg B_i \notin \Gamma_w$, for each $i \in \mathfrak{t}$, $\neg B_j \notin \Delta_w$, $B_j \notin \Delta_w$, for each $j \in \mathfrak{h}$, and $B_k \notin \Gamma_w$, for each $k \in \mathfrak{f}$. Using maximality of w and the fact that H^* is an $\mathbb{F}$-hypersequent, we obtain that $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg B_i, \Gamma_w \Rightarrow \Delta_w$, for each $i \in \mathfrak{t}$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, B_j$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \neg B_j$, for each $j \in \mathfrak{h}$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B_k, \Gamma_w \Rightarrow \Delta_w$, for each $k \in \mathfrak{f}$. Applying the rule $(\Rightarrow \circledcirc^{\mathbf{LPDG_3}}_{\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2})$, we obtain that $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \circledcirc(B_1, \ldots, B_n)$. By structural rules, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction.

Thus, $\neg B_i \in \Gamma_w$, for some $i \in \mathfrak{t}$, or $\neg B_j \in \Delta_w$, for some $j \in \mathfrak{h}$, $B_j \in \Delta_w$, for some $j \in \mathfrak{h}$, or $B_k \in \Gamma_w$, for some $k \in \mathfrak{f}$.

Suppose that $\neg B_i \in \Gamma_w$, for some $i \in \mathfrak{t}$. By the definition of the rule $(\Rightarrow \circledcirc^{\mathbf{LPDG_3}}_{\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2})$ and the set $\mathfrak{t}$, $v(B_i, w) = 1$. However, by the inductive hypothesis on B, $\mathsf{0} \in v(B_i, w)$, hence $v(B_i, w) \neq 1$. Contradiction.

Suppose that $\neg B_j \in \Delta_w$, for some $j \in \mathfrak{h}$. By the definition of the rule $(\Rightarrow \circledcirc^{\mathbf{LPDG_3}}_{\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2})$ and the set $\mathfrak{h}$, $v(B_i, w) = 1/2$. However, by the inductive hypothesis on B, $\mathsf{0} \notin v(B_i, w)$, hence $v(B_i, w) \neq 1/2$. Contradiction.

Suppose that $B_j \in \Delta_w$, for some $j \in \mathfrak{h}$. By the definition of the rule $(\Rightarrow \circledcirc^{\mathbf{LPDG_3}}_{\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2})$ and the set $\mathfrak{h}$, $v(B_i, w) = 1/2$. However, by the inductive hypothesis on B, $\mathsf{1} \notin v(B_i, w)$, hence $v(B_i, w) \neq 1/2$. Contradiction.

Suppose that $B_k \in \Gamma_w$, for some $k \in \mathfrak{f}$. By the definition of the rule $(\Rightarrow \circledcirc^{\mathbf{LPDG_3}}_{\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2})$ and the set $\mathfrak{f}$, $v(B_i, w) = 0$. However, by the inductive hypothesis on B, $\mathsf{1} \in v(B_i, w)$, hence $v(B_i, w) \neq 0$. Contradiction.

Therefore, $\mathsf{1} \notin v(\circledcirc(B_1, \ldots, B_n), w)$.

Suppose that $A \in \Gamma_w$, i.e., $\circledcirc(B_1, \ldots, B_n) \in \Gamma_w$. Similarly to the previous case, use $(\circledcirc \Rightarrow^{\mathbf{LPDG_3}}_{\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 0})$.

Suppose that $\neg A \in \Delta_w$, i.e., $\neg\circledcirc(B_1, \ldots, B_n) \in \Delta_w$. Similarly to the previous case, use the rule $(\Rightarrow \neg\circledcirc^{\mathbf{LPDG_3}}_{\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1/2})$.

Suppose that $\neg A \in \Gamma_w$, i.e., $\neg\circledcirc(B_1, \ldots, B_n) \in \Gamma_w$. Similarly to the previous case, use the rule $(\neg\circledcirc \Rightarrow^{\mathbf{LPDG_3}}_{\langle \mathfrak{t},\mathfrak{h},\mathfrak{f}\rangle 1})$.

Let A be $\rhd B$. Suppose that $A \in \Gamma_w$. Similarly to Theorem 6. Assume that there is $y \in W$ such that $B \notin \Gamma_y$ and there is a $z \in W$ such that $B \notin \Delta_z$. Since y and z are maximal and H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B, \Gamma_y \Rightarrow \Delta_y$ and $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_z \Rightarrow \Delta_z, B$. By the rule $(\rhd \Rightarrow)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \rhd B \Rightarrow \mid \Gamma_y \Rightarrow \Delta_y \mid \Gamma_z \Rightarrow \Delta_z$. By structural rules, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Hence, for each $x \in W$, $B \in \Gamma_x$, or for each $x \in W$, $B \in \Delta_x$. By the induction hypothesis for B, for each $x \in W$, $\mathsf{1} \in \vartheta(B, x)$ or for each $x \in W$, $\mathsf{1} \notin \vartheta(B, x)$. Thus, $\mathsf{1} \in \vartheta(A, w)$.

Suppose that $A \in \Delta_w$. Similarly to Theorem 6. Assume that $B \Rightarrow \notin H^*$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B \Rightarrow$, since H^* is an $\mathbb{F}$-sequence. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B \mid B \Rightarrow$. By $(\Rightarrow \rhd)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow \rhd B$. By structural rules, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Hence, $B \Rightarrow \in H^*$. Therefore, there is a $y \in W$ such that $B \in \Gamma_y$. By the induction hypothesis for B, there is a $y \in W$ such that $\mathsf{1} \in \vartheta(B, y)$. In a similar way we show that $\Rightarrow B \in H^*$. Thus, there is a $z \in W$ such that $B \in \Delta_z$. By the induction hypothesis for B, there is a $z \in W$ such that $\mathsf{1} \notin \vartheta(B, z)$. Therefore, $\mathsf{1} \notin \vartheta(A, w)$.

Suppose that $\neg A \in \Gamma_w$. Assume that $\neg B \Rightarrow \notin H^*$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg B \Rightarrow$, since H^* is an $\mathbb{F}$-sequence. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow \neg B \mid \neg B \Rightarrow$. By $(\neg\rhd \Rightarrow)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg\rhd B \Rightarrow$. By structural rules, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Hence, $\neg B \Rightarrow \in H^*$. Therefore, there is a $y \in W$ such that $\neg B \in \Gamma_y$. By the induction hypothesis for B, there is a $y \in W$ such that $\mathsf{0} \in \vartheta(B, y)$. In a similar way we show that $\Rightarrow \neg B \in H^*$. Thus, there is a $z \in W$ such that $\neg B \in \Delta_z$. By the induction hypothesis for B, there is a $z \in W$ such that $\mathsf{0} \notin \vartheta(B, z)$. Therefore, $\mathsf{0} \in \vartheta(A, w)$.

Suppose that $\neg A \in \Delta_w$. Assume that there is $y \in W$ such that $\neg B \notin \Gamma_y$ and there is a $z \in W$ such that $\neg B \notin \Delta_z$. Since y and z are maximal and H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg B, \Gamma_y \Rightarrow \Delta_y$ and $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_z \Rightarrow \Delta_z, \neg B$. By the rule $(\Rightarrow \neg\rhd)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow \neg\rhd B \mid \Gamma_y \Rightarrow \Delta_y \mid \Gamma_z \Rightarrow \Delta_z$. By structural rules, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Hence, for each $x \in W$, $\neg B \in \Gamma_x$, or for each $x \in W$, $\neg B \in \Delta_x$. By the induction hypothesis for B, for each $x \in W$, $\mathsf{0} \in \vartheta(B, x)$ or for each $x \in W$, $\mathsf{0} \notin \vartheta(B, x)$. Thus, $\mathsf{0} \notin \vartheta(A, w)$.

Let A be $\circ B$. Suppose that $A \in \Gamma_w$. Suppose that $B \notin \Delta_w$ and for some maximal $x \in W$, $B \notin \Gamma_x$. Then by the maximality of w and x as well as the fact that H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, B$ and $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B, \Gamma_x \Rightarrow \Delta_x$. By the rule $(\circ \Rightarrow)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \circ B, \Gamma_w \Rightarrow \Delta_w, \mid \Gamma_x \Rightarrow \Delta_x$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid w \mid x$. By (Merge), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Thus, $B \in \Delta_w$ or for each $x \in W$, $B \in \Gamma_x$. It follows by the induction hypothesis for B that $\mathsf{1} \notin \vartheta(B, w)$ or for each maximal $x \in W$, $\mathsf{1} \in \vartheta(B, x)$. Hence, $\mathsf{1} \in \vartheta(A, w)$.

Suppose that $A \in \Delta_w$. Assume that $B \notin \Gamma_w$ or $\Rightarrow B \notin H^*$. Suppose that $B \notin \Gamma_w$. Then since w is maximal, $B, \Gamma_w \Rightarrow \Delta_w \notin H^*$. Since $B \notin \Gamma_w$ and H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid B, \Gamma_w \Rightarrow \Delta_w$. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B \mid B, \Gamma_w \Rightarrow \Delta_w$. By the rule $(\Rightarrow \circ)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \circ B$. Since $A \in \Delta_w$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid w$. Using (Merge), we have $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Suppose that $\Rightarrow B \notin H^*$. Since H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B$. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow B \mid B, \Gamma_w \Rightarrow \Delta_w$. Using $(\Rightarrow \circ)$ and (Merge), we get $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Hence, $B \in \Gamma_w$ and $\Rightarrow B \in H^*$. Then by the induction hypothesis for B, $\mathsf{1} \in \vartheta(B, w)$ and for some $x \in W$, $\mathsf{1} \notin \vartheta(B, x)$. Hence, $\mathsf{1} \notin \vartheta(A, w)$.

Suppose that $\neg A \in \Gamma_w$. Assume that $\neg B \notin \Delta_w$ or $\neg B \Rightarrow \notin H^*$. Suppose that $\neg B \notin \Delta_w$. Then since w is maximal, $\Gamma_w \Rightarrow \Delta_w, \neg B \notin H^*$. Since H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \neg B$. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg B \Rightarrow \mid \Gamma_w \Rightarrow \Delta_w, \neg B$. By the rule $(\Rightarrow \circ)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg\circ B, \Gamma_w \Rightarrow \Delta_w$. Since $A \in \Delta_w$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid w$. Using (Merge), we have $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Suppose that $\neg B \Rightarrow \notin H^*$. Since H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg B \Rightarrow$. By (EW), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg B \Rightarrow \mid \Gamma_w \Rightarrow \Delta_w, \neg B$. Using $(\neg\circ \Rightarrow)$ and (Merge), we get $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Hence, $\neg B \in \Delta_w$ and $\neg B \Rightarrow \in H^*$. Then by the induction hypothesis for B, $\mathsf{0} \notin \vartheta(B, w)$ and for some $x \in W$, $\mathsf{0} \in \vartheta(B, x)$. Hence, $\mathsf{0} \in \vartheta(A, w)$.

Let A be $\circ B$. Suppose that $\neg A \in \Delta_w$. Suppose that $\neg B \notin \Gamma_w$ and for some maximal $x \in W$, $\neg B \notin \Delta_x$. Then by the maximality of w and x as well as the fact that H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \neg B, \Gamma_w \Rightarrow \Delta_w$ and $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Gamma_x \Rightarrow \Delta_x, \neg B$. By the rule $(\Rightarrow \neg\circ)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \circ\Gamma_w \Rightarrow \Delta_w, \neg\circ B \mid \Gamma_x \Rightarrow \Delta_x$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid w \mid x$. By (Merge), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Thus, $\neg B \in \Gamma_w$ or for each $x \in W$, $\neg B \in \Delta_x$. It follows by the induction hypothesis for B that $\mathsf{0} \in \vartheta(B, w)$ or for each maximal $x \in W$, $\mathsf{0} \notin \vartheta(B, x)$. Hence, $\mathsf{0} \notin \vartheta(A, w)$.

The other cases are considered similarly.

Finally, we need to show that $\langle W, \vartheta \rangle$ is a model for $\mathscr{H}$, but not for H. It can be done analogously to the similar claim from the proof of Theorem 6. $\square$

Corollary 144. *Let* $\mathbf{L} \in \{\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}, \mathbf{LP}_{\neg}^{i(\circledcirc)_m}, \mathbf{FDE}_{\neg}^{j(\circledcirc)_m}\}$, *where* $1 \leqslant i \leqslant 4$, $1 \leqslant j \leqslant 16$. *Let* $\clubsuit \in \{\Box, \Diamond, \rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$ *and* $\mathbf{L} = \mathbf{S5}^{\clubsuit}$. *For each finite set of hypersequents* $\mathscr{H} \cup \{H\}$, $\mathscr{H} \vdash_{\mathbb{HSL}} H$ *iff* $\mathscr{H} \models_{\mathbf{L}} H$.

Proof. Follows from Theorems 142 and 143. $\square$

Corollary 145. *Let* $\mathbf{L} \in \{\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}, \mathbf{LP}_{\neg}^{i(\circledcirc)_m}, \mathbf{FDE}_{\neg}^{j(\circledcirc)_m}\}$, *where* $1 \leqslant i \leqslant 4$, $1 \leqslant j \leqslant 16$. *Let* $\clubsuit \in \{\Box, \Diamond, \rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$ *and* $\mathbf{L} = \mathbf{S5}^{\clubsuit}$. *Then* $\mathscr{H} \vdash_{\mathbb{HSL}} H$ *implies* $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H$.

Proof. Follows from Theorem 143. Notice that in the proof of this theorem, (Cut) is used only once in order to show that $\langle W, \vartheta \rangle$ is a model for $\mathscr{H}$ and is applied only to formulas which belong to $\mathscr{H}$. $\square$

Corollary 146 (Cut admissibility). *Let* $\mathbf{L} \in \{\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}, \mathbf{LP}_{\neg}^{i(\circledcirc)_m}, \mathbf{FDE}_{\neg}^{j(\circledcirc)_m}\}$*, where* $1 \leqslant i \leqslant 4$*,* $1 \leqslant j \leqslant 16$*. Let* $\clubsuit \in \{\Box, \Diamond, \rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$ *and* $\mathbf{L} = \mathbf{S5}^{\clubsuit}$*. Then* $\vdash_{\mathbb{HSL}} H$ *implies that there is a cut-free proof of* H *in* $\mathbb{HS}\mathbf{L}$*.*

Proof. Put $\mathscr{H} = \emptyset$ in the proof of Theorem 143. Then the only application of (Cut) in the proof of this Theorem disappears. $\square$

Corollary 147 (Negated subformula property). *Let* $\mathbf{L} \in \{\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}, \mathbf{LP}_{\neg}^{i(\circledcirc)_m}, \mathbf{FDE}_{\neg}^{j(\circledcirc)_m}\}$*, where* $1 \leqslant i \leqslant 4$*,* $1 \leqslant j \leqslant 16$*. Let* $\clubsuit \in \{\Box, \Diamond, \rhd, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$ *and* $\mathbf{L} = \mathbf{S5}^{\clubsuit}$*. For every hypersequent which is provable in* $\mathbb{HS}\mathbf{L}$*, there is a proof such that each formula which occurs in it is a negated subformula of the formulas which occur in the conclusion.*

Proof. Follows from Corollary 146 and the fact that in any of the rules of $\mathbb{HS}\mathbf{L}$ each formula which occurs in the premises is a subformula of the formulas which occur in the conclusion. $\square$

4.3 Nested sequent calculi for modal many-valued logics

For the case of modal many-valued logics weaker than **S5**, we need to use the nested sequent calculi framework. Nested sequent calculi for modal many-valued logics are obtained from the nested sequent calculi for two-valued modal logics by replacing the classical rules with the rules (in their nested sequent formulations) for three- and four-valued logics considered in the previous chapter and adding the rules for the negated modalities which are given below.

$$[\neg\Box \Rightarrow]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \neg A \Rightarrow]}{\mathfrak{N}[\neg\Box A, \Gamma \Rightarrow \Delta]} \qquad [\Rightarrow \neg\Box]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg\Box A / (\Theta \Rightarrow \Lambda, \neg A / X)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg\Box A / (\Theta \Rightarrow \Lambda / X)]}$$

$$[\neg\Diamond \Rightarrow]\ \frac{\mathfrak{N}[\neg\Diamond A, \Gamma \Rightarrow \Delta / (\neg A, \Theta \Rightarrow \Lambda / X)]}{\mathfrak{N}[\neg\Diamond A, \Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / X)]} \qquad [\Rightarrow \neg\Diamond]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \Rightarrow \neg A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg\Diamond A]}$$

$$[\neg\rhd \Rightarrow_L]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \neg A \Rightarrow]}{\mathfrak{N}[\neg\rhd A, \Gamma \Rightarrow \Delta]} \qquad [\neg\rhd \Rightarrow_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \Rightarrow \neg A]}{\mathfrak{N}[\neg\rhd A, \Gamma \Rightarrow \Delta]}$$

$$[\Rightarrow \neg\rhd]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg\rhd A / (\neg A, \Theta \Rightarrow \Lambda / X)] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta, \neg\rhd A / (\Xi \Rightarrow \Pi, \neg A / Y)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg\rhd A / (\Theta \Rightarrow \Lambda / X); (\Xi \Rightarrow \Pi / Y)]}$$

$$[\neg\blacktriangleright \Rightarrow]\ \frac{\mathfrak{N}[\neg\blacktriangleright A, \Gamma \Rightarrow \Delta / (\neg A, \Theta \Rightarrow \Lambda / X)] \quad \mathfrak{N}[\neg\blacktriangleright A, \Gamma \Rightarrow \Delta / (\Xi \Rightarrow \Pi, \neg A / Y)]}{\mathfrak{N}[\neg\blacktriangleright A, \Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda / X); (\Xi \Rightarrow \Pi / Y)]}$$

$$[\Rightarrow \neg\blacktriangleright_L]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \neg A \Rightarrow]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg\blacktriangleright A]} \qquad [\Rightarrow \neg\blacktriangleright_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \Rightarrow \neg A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg\blacktriangleright A]}$$

$$[\neg\circ \Rightarrow_L]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \neg A \Rightarrow]}{\mathfrak{N}[\neg\circ A, \Gamma \Rightarrow \Delta]} \qquad [\neg\circ \Rightarrow_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg A]}{\mathfrak{N}[\neg\circ A, \Gamma \Rightarrow \Delta]}$$

$$[\Rightarrow \neg\circ]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg\circ A / (\Theta \Rightarrow \Lambda, \neg A / X)] \quad \mathfrak{N}[\neg A, \Xi \Rightarrow \Pi, \neg\circ A / Y]}{\mathfrak{N}[\Gamma, \Xi \Rightarrow \Delta, \Pi, \neg\circ A / Y; (\Theta \Rightarrow \Lambda / X)]}$$

$$[\neg\bullet \Rightarrow]\ \frac{\mathfrak{N}[\neg\bullet A, \Gamma \Rightarrow \Delta / (\Theta \Rightarrow \Lambda, \neg A / X)] \quad \mathfrak{N}[\neg\bullet A, \neg A, \Xi \Rightarrow \Pi / Y]}{\mathfrak{N}[\neg\bullet A, \Gamma, \Xi \Rightarrow \Delta, \Pi / Y; (\Theta \Rightarrow \Lambda / X)]}$$

$$[\Rightarrow\neg\bullet_L]\ \dfrac{\mathfrak{N}[\Gamma\Rightarrow\Delta/\neg A\Rightarrow]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,\neg\bullet A]} \qquad [\Rightarrow\neg\bullet_R]\ \dfrac{\mathfrak{N}[\Gamma\Rightarrow\Delta,\neg A]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,\neg\bullet A]}$$

$$[\neg\widetilde{\circ}\Rightarrow_L]\ \dfrac{\mathfrak{N}[\neg A,\Gamma\Rightarrow\Delta]}{\mathfrak{N}[\neg\widetilde{\circ}A,\Gamma\Rightarrow\Delta]} \qquad [\neg\widetilde{\circ}\Rightarrow_R]\ \dfrac{\mathfrak{N}[\Gamma\Rightarrow\Delta/\Rightarrow\neg A]}{\mathfrak{N}[\neg\widetilde{\circ}A,\Gamma\Rightarrow\Delta]}$$

$$[\Rightarrow\neg\widetilde{\circ}]\ \dfrac{\mathfrak{N}[\Theta\Rightarrow\Lambda,\neg A,\neg\widetilde{\circ}A/X] \quad \mathfrak{N}[\Gamma\Rightarrow\Delta/(\neg A,\Xi\Rightarrow\Pi,\neg\widetilde{\circ}A/Y)]}{\mathfrak{N}[\Gamma,\Theta\Rightarrow\Delta,\Lambda,\neg\widetilde{\circ}A/X;(\Xi\Rightarrow\Pi/Y)]}$$

$$[\Rightarrow\neg\widetilde{\bullet}_L]\ \dfrac{\mathfrak{N}[\Gamma\Rightarrow\Delta/\Rightarrow\neg A]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,\neg\widetilde{\bullet}A]} \qquad [\Rightarrow\neg\widetilde{\bullet}_R]\ \dfrac{\mathfrak{N}[\neg A,\Gamma\Rightarrow\Delta]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,\neg\widetilde{\bullet}A]}$$

$$[\neg\widetilde{\bullet}\Rightarrow]\ \dfrac{\mathfrak{N}[\neg\widetilde{\bullet}A,\Theta\Rightarrow\Lambda,\neg A/X] \quad \mathfrak{N}[\neg\widetilde{\bullet}A,\Gamma\Rightarrow\Delta/(\neg A,\Xi\Rightarrow\Pi/Y)]}{\mathfrak{N}[\neg\widetilde{\bullet}A,\Gamma,\Theta\Rightarrow\Delta,\Lambda/X;(\Xi\Rightarrow\Pi/Y)]}$$

$$[\neg{\sim}\Rightarrow]\ \dfrac{\mathfrak{N}[\neg{\sim}A,\Gamma\Rightarrow\Delta/(\neg A,\Theta\Rightarrow\Lambda/X)]}{\mathfrak{N}[\neg{\sim}A,\Gamma\Rightarrow\Delta/(\Theta\Rightarrow\Lambda/X)]} \qquad [\Rightarrow\neg{\sim}]\ \dfrac{\mathfrak{N}[\Gamma\Rightarrow\Delta/\Rightarrow\neg A]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,\neg{\sim}A]}$$

$$[\neg\dot{\sim}\Rightarrow]\ \dfrac{\mathfrak{N}[\Gamma\Rightarrow\Delta/\neg A\Rightarrow]}{\mathfrak{N}[\neg\dot{\sim}A,\Gamma\Rightarrow\Delta]} \qquad [\Rightarrow\neg\dot{\sim}]\ \dfrac{\mathfrak{N}[\Gamma\Rightarrow\Delta,\neg\dot{\sim}A/(\Theta\Rightarrow\Lambda,\neg A/X)]}{\mathfrak{N}[\Gamma\Rightarrow\Delta,\neg\dot{\sim}A/(\Theta\Rightarrow\Lambda/X)]}$$

Theorem 148. *Let* $\mathbf{Y}\in\{\mathbf{K}_{\mathbf{3}_\neg}^{i(\circledcirc)_m},\mathbf{LP}_\neg^{i(\circledcirc)_m},\mathbf{FDE}_\neg^{j(\circledcirc)_m}\}$*, where* $1\leqslant i\leqslant 4$*,* $1\leqslant j\leqslant 16$*. Let* $\clubsuit\in\{\Box,\Diamond,\rhd,\blacktriangleright,\circ,\bullet,\widetilde{\circ},\widetilde{\bullet},\sim,\dot{\sim}\}$*. Let* $\mathbf{X}_1,\ldots\mathbf{X}_m\in\{\mathbf{T},\mathbf{D},\mathbf{4},\mathbf{B}\}$*. For any nested sequent* $\mathfrak{N}$*, if* $\mathbb{NS}\mathbf{KX}_1,\ldots\mathbf{X}_m\mathbf{Y}^\clubsuit\vdash\mathfrak{N}$*, then* $\mathbf{KX}_1,\ldots\mathbf{X}_m\mathbf{Y}^\clubsuit\models\mathfrak{N}$*.*

Proof. Similarly to Theorem 27. □

Theorem 149. *Let* $\mathbf{Y}\in\{\mathbf{K}_{\mathbf{3}_\neg}^{i(\circledcirc)_m},\mathbf{LP}_\neg^{i(\circledcirc)_m},\mathbf{FDE}_\neg^{j(\circledcirc)_m}\}$*, where* $1\leqslant i\leqslant 4$*,* $1\leqslant j\leqslant 16$*. Let* $\clubsuit\in\{\Box,\Diamond,\rhd,\blacktriangleright,\circ,\bullet,\widetilde{\circ},\widetilde{\bullet},\sim,\dot{\sim}\}$*. Let* $\mathbf{X}_1,\ldots\mathbf{X}_m\in\{\mathbf{T},\mathbf{D},\mathbf{4},\mathbf{B}\}$*. For any nested sequent* $\mathfrak{N}$*, if* $\mathbb{NS}\mathbf{KX}_1,\ldots\mathbf{X}_m\mathbf{Y}^\clubsuit\vdash\mathfrak{N}$*, then* $\mathbf{KX}_1,\ldots\mathbf{X}_m\mathbf{Y}^\clubsuit\models\mathfrak{N}$*.*

Proof. Similarly to Theorem 36. □

Theorem 150 (Cut admissibility)**.** *Let* $\mathbf{Y}\in\{\mathbf{K}_{\mathbf{3}_\neg}^{i(\circledcirc)_m},\mathbf{LP}_\neg^{i(\circledcirc)_m},\mathbf{FDE}_\neg^{j(\circledcirc)_m}\}$*, where* $1\leqslant i\leqslant 4$*,* $1\leqslant j\leqslant 16$*. Let* $\clubsuit\in\{\Box,\Diamond,\rhd,\blacktriangleright,\circ,\bullet,\widetilde{\circ},\widetilde{\bullet},\sim,\dot{\sim}\}$*.* $\mathbf{L}=\mathbb{NS}\mathbf{KY}^\clubsuit$ *and* $\mathfrak{N}$ *be a nested sequent. Then* $\vdash_{\mathbb{NS}\mathbf{L}}\mathfrak{N}$ *implies that there is a cut-free proof of* $\mathfrak{N}$ *in* $\mathbb{NS}\mathbf{L}$*.*

Proof. Follows from Theorem 149 and the fact that in its proof the rule of cut is not needed for its proof. □

4.4 Natural deduction for modal many-valued logics

The rules for negated $\Box$ and $\Diamond$ can be easily obtained from the rules for $\sim$ and $\dot{\sim}$ (recall that ${\sim}A=\neg\Box A$ and $\dot{\sim}A=\neg\Diamond A$) from Section 2.5.

$$(\neg\Box GI)^a\ \dfrac{\begin{array}{cc}[\neg\Box A]^a & \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ F & \neg A\end{array}}{F} \qquad (\neg\Box E)^a\ \dfrac{\begin{array}{ccc} & & [\neg A,B_1\ldots B_m]^a \\ \mathfrak{D}_0 & \mathfrak{D}_1\ldots\mathfrak{D}_m & \mathfrak{E} \\ \neg\Box A & B_1\ldots B_m & C\end{array}}{C}$$

$$(\neg\Diamond GI)^{a,b}\ \dfrac{\begin{array}{ccc}[\neg\Diamond A]^a & & [B_1\ldots B_m]^b \\ \mathfrak{D}_0 & \mathfrak{D}_1\ldots\mathfrak{D}_m & \mathfrak{E} \\ F & B_1\ldots B_m & \neg A\end{array}}{F} \qquad (\neg\Diamond GE)^a\ \dfrac{\begin{array}{cc} & [\neg A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg\Diamond A & F\end{array}}{F}$$

$B_1, \dots, B_m$ are exactly the undischarged assumptions in $\mathfrak{D}$ and $\neg A, B_1, \dots, B_m$ in $\mathfrak{E}$. For **S5**-style logics, $\neg A, B_1, \dots, B_m, C$ are required to be modalized.

We can also offer the rules for the negated operators $\sim$ and $\dot{\sim}$, taking into account that $\neg{\sim}A = \Box A$ and $\neg\dot{\sim}A = \Diamond A$.

$$(\neg{\sim}GI)^{a,b}\ \dfrac{\begin{array}{c}[\neg{\sim}A]^a\\ \mathfrak{D}_0\\ F\end{array}\quad \begin{array}{c}\mathfrak{D}_1\dots\mathfrak{D}_m\\ B_1\dots B_m\end{array}\quad \begin{array}{c}[B_1\dots B_m]^b\\ \mathfrak{D}\\ A\end{array}}{F}\qquad (\neg{\sim}GE)^a\ \dfrac{\begin{array}{c}\mathfrak{D}_1\\ \neg{\sim}A\end{array}\quad \begin{array}{c}[A]^a\\ \mathfrak{D}_2\\ F\end{array}}{F}$$

$$(\neg\dot{\sim}GI)^a\ \dfrac{\begin{array}{c}[\neg\dot{\sim}A]^a\\ \mathfrak{D}_1\\ F\end{array}\quad \begin{array}{c}\mathfrak{D}_2\\ A\end{array}}{F}\qquad (\neg\dot{\sim}E)^a\ \dfrac{\begin{array}{c}\mathfrak{D}_0\\ \neg\dot{\sim}A\end{array}\quad \begin{array}{c}\mathfrak{D}_1\dots\mathfrak{D}_m\\ B_1\dots B_m\end{array}\quad \begin{array}{c}[A, B_1\dots B_m]^a\\ \mathfrak{E}\\ C\end{array}}{C}$$

$B_1, \dots, B_m$ are exactly the undischarged assumptions in $\mathfrak{D}$ in $(\Box I)$ and $A, B_1, \dots, B_m$ in $\mathfrak{E}$ in $(\Diamond E)$. For **S5**-style logics, $A, B_1, \dots, B_m, C$ are required to be modalized.

As for normalisation, one should take as a basis the normalisation proof for many-valued logics (with all definitions, such as degree of a formula, etc., all reduction and permutation procedures, inductions, and so on), and then add there the cases for modalities from Chapter 1, additional cases for negated modalities, and the cases produced by many-valued and modal rules at the same time. These cases are treated similarly to all the previous ones; let us give some examples.

Case (Example) 1. The maximal formula $\neg\Box A$ is produced by applications of the rules $(\neg\Box E)$ and $(\neg\Box GI)$.

$$(\neg\Box GI)^a\ \dfrac{(\neg\Box E)^b\ \dfrac{[\neg\Box A]^a\quad \begin{array}{c}\mathfrak{D}_1\dots\mathfrak{D}_m\\ B_1\dots B_m\end{array}\quad \begin{array}{c}[\neg A, B_1\dots B_m]^b\\ \mathfrak{E}_1\\ C\end{array}}{C}\qquad \begin{array}{c}\mathfrak{E}_2\\ \neg A\end{array}}{C}\quad \dashrightarrow\quad (\neg\Box GI)^a\ \dfrac{\begin{array}{c}\mathfrak{E}_2\ \mathfrak{D}_1\dots\mathfrak{D}_m\\ \neg A\ \ B_1\dots B_m\\ \mathfrak{E}_1\\ C\end{array}\qquad \begin{array}{c}\mathfrak{E}_2\\ \neg A\end{array}}{C}$$

Case (Example) 2. The maximal segment with the formula $\circledcirc(\vec{F})$ is produced by applications of the rules $R(\langle\mathfrak{t},\mathfrak{f}\rangle 0)$ and $(\Box GE)$. $\mathfrak{R}_1$ and $\mathfrak{R}_2$ stand for $R(\langle\mathfrak{t},\mathfrak{f}\rangle 0)$ and $(\Box GE)$, respectively.

$$\mathfrak{R}_2^{b,c}\ \dfrac{\mathfrak{R}_1^a\ \dfrac{\begin{array}{c}\mathfrak{D}_0\\ \Box A\end{array}\quad \begin{array}{c}[A]^a\\ \mathfrak{D}_1\\ \circledcirc(\vec{F})\end{array}}{\circledcirc(\vec{F})}\quad \begin{array}{c}[\neg F_i^\dagger]^b\\ \mathfrak{E}_1^\dagger\\ G\end{array}\quad \begin{array}{c}\mathfrak{E}_2^\ddagger\\ F_j^\ddagger\end{array}\quad \begin{array}{c}\mathfrak{E}_3^\ddagger\\ \neg F_j^\ddagger\end{array}\quad \begin{array}{c}[F_k^*]^c\\ \mathfrak{E}_4^*\\ G\end{array}}{G}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

We transform it as follows:

$$\mathfrak{R}_1^a\ \dfrac{\begin{array}{c}\mathfrak{D}_0\\ \Box A\end{array}\quad \mathfrak{R}_2^{b,c}\ \dfrac{\begin{array}{c}[A]^a\\ \mathfrak{D}_1\\ \circledcirc(\vec{F})\end{array}\quad \begin{array}{c}[\neg F_i^\dagger]^b\\ \mathfrak{E}_1^\dagger\\ G\end{array}\quad \begin{array}{c}\mathfrak{E}_2^\ddagger\\ F_j^\ddagger\end{array}\quad \begin{array}{c}\mathfrak{E}_3^\ddagger\\ \neg F_j^\ddagger\end{array}\quad \begin{array}{c}[F_k^*]^c\\ \mathfrak{E}_4^*\\ G\end{array}}{G}}{G}$$

† for each $i \in \mathfrak{t}$, ‡ for each $j \in \mathfrak{h}$, * for each $k \in \mathfrak{f}$.

Theorem 151. *Let* $\clubsuit \in \{\Box, \Diamond, \triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$. *Let* $\mathbf{X} \in \{\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}, \mathbf{LP}_{\neg}^{i(\circledcirc)_m}, \mathbf{FDE}_{\neg}^{j(\circledcirc)_m}\}$, *where* $1 \leqslant i \leqslant 4$, $1 \leqslant j \leqslant 16$. *Let* $\mathbf{L} = \mathbf{XS5}^{\clubsuit}$. *Any deduction in* $\mathbf{L}$ *can be converted into a deduction in normal form.*

Proof. Similarly to Theorem 100. □

Theorem 152. *Let* $\clubsuit \in \{\Box, \Diamond, \triangleright, \blacktriangleright, \circ, \bullet, \widetilde{\circ}, \widetilde{\bullet}, \sim, \dot{\sim}\}$. *Let* $\mathbf{X} \in \{\mathbf{K}_{\mathbf{3}\neg}^{i(\circledcirc)_m}, \mathbf{LP}_{\neg}^{i(\circledcirc)_m}, \mathbf{FDE}_{\neg}^{j(\circledcirc)_m}\}$, *where* $1 \leqslant i \leqslant 4$, $1 \leqslant j \leqslant 16$. *Let* $\mathbf{L} = \mathbf{XS5}^{\clubsuit}$. *Deductions in normal forms in* $\mathbf{L}$ *have the negation subformula property.*

Proof. Similarly to Theorem 104. □

4.5 Proof systems for modal multilattice logics

Multilattice logic[2] can be viewed as an algebraic generalisation of some many-valued logics. It generalises Arieli and Avron's four-valued bilattice logic [2], Shramko and Wansing's sixteen-valued trilattice logic [179], Zaitsev's eight-valued tetralattice logic [198]. These logics themselves generalise Belnap-Dunn's four-valued logic **FDE** (First Degree Entailment) [13, 14, 34]. From an algebraic point of view, multilattices (or n-lattices) generalise bilattices [64, 65], trilattices [178], and tetralattices [198] which themselves generalise De Morgan lattices, that is an algebraic semantics of **FDE**.

Multilattice logic $\mathbf{ML}_n$ was first formulated by Shramko in [177], later on, by Kamide and Shramko [91] it was extended by implications and co-implications as well as a first-order version $\mathbf{FML}_n$ was presented. Among other multilattice logics are Kamide, Shramko, and Wansing's bi-intuitionistic multilattice logic $\mathbf{BML}_n$ and its connexive variant $\mathbf{CML}_n$ [93], Kamide's linear multilattice logics $\mathbf{EML}_n$ and $\mathbf{LML}_n$ [89], Kamide's submultilattice logic $\mathbf{SM}_n$ and indexed multilattice logic $\mathbf{IM}_n$ [87], a fragment of $\mathbf{ML}_n$, called $\mathbf{MLL}_n$, determined by logical multilattices ($\mathbf{ML}_n$ itself is determined by ultralogical multilattices) studied by Grigoriev and the author [68], and several modal multilattice logics which are the subject of our interest. Modal multilattice logic $\mathbf{MML}_n$ was introduced by Kamide and Shramko [92], it has **S4**-style modalities, but without an interdefinability of necessity and possibility operators. A version of $\mathbf{MML}_n$ with the interdefinability was proposed by Grigoriev and the author [70] under the name $\mathbf{MML}_n^{\mathbf{S4}}$. Kamide and Shramko [92] posed the problem of the formulation of modal multilattice multilattice logic with **S5**-style modalities, such logic, called $\mathbf{MML}_n^{\mathbf{S5}}$, was developed by Grigoriev and the author in [71]. Although Kamide and Shramko formulated the notion of modal multilattice (multilattice with Tarski's interior and closure operators), they did not provide an algebraic completeness proof for $\mathbf{MML}_n$. Besides, as follows from Cattaneo and Ciucci's research [24] to which Kamide and Shramko refer, **S4**-style modalities require Kuratowski's interior and closure operators, while Tarski's ones correspond to a weaker logic, **MNT4** (for **S5** Halmos' operators are needed). Also, the original notion of modal multilattice lacks some postulates regarding inversions of closure and interior operators. An improved notion of modal multilattice was given by Grigoriev and the author in [70] (called De Morgan modal multilattice), $\mathbf{MML}_n$ was proven to be sound and complete with respect to it; in [69], the notions of Tarski, Kuratowski, and Halmos multilattices were formulated as well as a new logic, $\mathbf{MML}_n^{\mathbf{MNT4}}$, a modal multilattice logic with **MNT4**-style modalities, was described. As follows from [69], $\mathbf{MML}_n^{\mathbf{MNT4}}$, $\mathbf{MML}_n^{\mathbf{S4}}$, and $\mathbf{MML}_n^{\mathbf{S5}}$, respectively, are sound and complete with respect to Tarski, Kuratowski, and Halmos multilattices. Later on in [68], there were studied some congruent and monotonic modal multilattice logics, which are fragments of $\mathbf{MML}_n^{\mathbf{MNT4}}$.

[2]This section is based on the results obtained by the author in coauthorship with Grigoriev in [71, 69] as well as some new previously unpublished results. In particular, the notions of Tarski, Kuratowski, and Halmos are taken from [69]; the formulation of the hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$ with standard modalities is the result from [71]. The author and Oleg Grigoriev have contributed equally to the papers [71, 69].

In this section, we plan to mainly consider logic $\mathbf{MML}_n^{\mathbf{S5}}$, and partly logic $\mathbf{MML}_n^{\mathbf{S4}}$. Also, we briefly describe how other modal multilattice logics can be obtained and formalised via nested sequent calculi. We are going to describe some results obtained by Grigoriev and the author [71, 69, 70] and some new ones. The cut-free hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$ developed in [71] is based on Restall's hypersequent calculus for **S5** and is a more general version of an adaptation of Restall's calculus for three- and four-valued versions of **S5** considered in previous sections. In [71], soundness, completeness, and cut admissibility were established by the technique of embedding theorems. In this section, we give a new completeness proof with respect to Kripke semantics by the Hintikka-style method that is very similar to proofs that were presented before for **S5**-style non-standard and many-valued modalities. Additionally, we present a constructive cut elimination proof similar to the proofs considered in the previous sections. Last but not least, we formulate a multilattice version of non-standard modalities.

Although our main hero is $\mathbf{MML}_n^{\mathbf{S5}}$, since it is a good example of modal multilattice logic and is closely connected to the results from the previous sections, we will pay a bit more attention to $\mathbf{MML}_n^{\mathbf{S4}}$. Although one may find a cut-free sequent calculus for it in [69], it will be interesting for us from the point of view of natural deduction. Since the approach to natural deduction for modal logics that we use in this paper allows us to obtain natural deduction systems for both **S4**- and **S5**-style logics, we will introduce natural deduction systems for both $\mathbf{MML}_n^{\mathbf{S4}}$ and $\mathbf{MML}_n^{\mathbf{S5}}$ (and $\mathbf{ML}_n$ itself) which is another new result about these logics. Despite the fact that $\mathbf{ML}_n$ generalizes many-valued logics, it contains the Boolean negation. As a result, the natural deduction system has some common features with the systems of classical logic. In the case of multilattice logic, we do not consider n-ary connectives: just negations, conjunctions, and disjunctions corresponding to inversions, meet and join operations of a multilattice (also implications and coimplications can be expressed via them). So we do not have a Segerberg-style natural deduction system for multilattice logic. But we are able to present a system in the style of Milne's [125] natural deduction system for classical logic with general introduction and elimination rules (which has a constructive proof of normalisation due to Kürbis [103]). We prove the normalisation theorem for $\mathbf{MML}_n^{\mathbf{S4}}$ and $\mathbf{MML}_n^{\mathbf{S5}}$ (although for simplicity we consider the language without (co)implications and with just one type of modal operators: either necessity or possibility for $\mathbf{MML}_n^{\mathbf{S5}}$ and only necessity for $\mathbf{MML}_n^{\mathbf{S4}}$).

Besides, we formulate via Kripke semantics the modal multilattice logic $\mathbf{MML}_n^{\mathbf{K}}$ and its reflexive, serial, transitive, and symmetric extensions. Then we present cut-free sound and complete nested sequent calculi for them, using methods similar to the ones that we applied in Section 2.4.

Now let us begin with some preliminaries about lattices, following their presentation in [35].

Definition 153 (Lattice)**.** A *lattice* is a structure $\langle L, \cap, \cup \rangle$ with the relation $a \leqslant b$ defined as $a \cap b = a$. Postulates characterising the operations are as follows, for each $a, b \in L$:

- Idempotence: $a \cap a = a$, $a \cup a = a$
- Commutativity: $a \cap b = b \cap a$, $a \cup b = b \cup a$
- Associativity: $a \cap (b \cap c) = (a \cap b) \cap c$, $a \cup (b \cup c) = (a \cup b) \cup c$
- Absorption: $a \cap (a \cup b) = a$, $a \cup (a \cap b) = a$.

Definition 154 (Distributive lattice)**.** $\langle L, \cap, \cup \rangle$ is a *distributive* lattice iff it is a lattice satisfying the following postulate, for any $a, b, c \in L$: $a \cap (b \cup c) \leqslant (a \cap b) \cup c$.

Definition 155 (De Morgan lattice)**.** $\langle L, \cap, \cup, - \rangle$ is a *de Morgan lattice* iff $\langle L, \cap, \cup \rangle$ is a distributive lattice and $-$ satisfies the following conditions, for each $a \in L$:

- $a \leqslant b$ iff $-b \leqslant -a$,
- $a = --a$.

Multilattices generalize De Morgan lattices. We are ready now to present the notion of a multilattice and some other important related notions, following their description in [177, 70, 69].

Definition 156. [177, p. 204, Definition 4.1] A *multilattice* is a structure $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$, where $n > 1$, $S \neq \emptyset$, $\leqslant_1, \ldots, \leqslant_n$ are partial orders such that $\langle S, \leqslant_1 \rangle, \ldots, \langle S, \leqslant_n \rangle$ are lattices with the corresponding pairs of meet and join operators $\langle \cap_1, \cup_1 \rangle, \ldots, \langle \cap_n, \cup_n \rangle$ as well as the corresponding j-inversion operators $-_1, \ldots, -_n$ which satisfy the following conditions, for each $j, k \leqslant n$, $j \neq k$, and $a, b \in S$:

$$a \leqslant_j b \text{ implies } -_j b \leqslant_j -_j a; \tag{anti}$$

$$a \leqslant_k b \text{ implies } -_j a \leqslant_k -_j b; \tag{iso}$$

$$-_j -_j a = a. \tag{per2}$$

Remark 157. Notice that an algebraic completeness proof for a multilattice logic $\mathbf{ML}_n$ and its modal extensions given in [70] (see [69, 68] for modal extensions) uses a bit different notion of a multilattice: the conditions (anti) and (iso) are replaced by the following ones:

$$-_j(a \cap_j b) = -_j a \cup_j -_j b; \tag{DM1}$$

$$-_j(a \cup_j b) = -_j a \cap_j -_j b; \tag{DM2}$$

$$-_k(a \cap_j b) = -_k a \cap_j -_k b; \tag{DM3}$$

$$-_k(a \cup_j b) = -_k a \cup_j -_k b. \tag{DM4}$$

A more detailed exposition of algebraic completeness of multilattice logics can be found in [70, 69, 68].

Definition 158 (Distributive multilattice)**.** [92, p. 319, Definition 2.1]. A multilattice $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ is called *distributive* iff all $2(2n^2 - n)$ distributive laws are satisfied, i.e., $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$, where $a, b, c \in S$, $\otimes, \oplus \in \{\cup_1, \cap_1, \ldots, \cup_n, \cap_n\}$, and $\otimes \neq \oplus$.

Remark 159. Although the subsequent definitions in this Section do not require distributivity, we are going to deal with *distrubutive* multilattices exclusively in our research.

Definition 160 (Multifilter)**.** [92, p. 319, Definition 2.3] Let $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ be a multilattice. $\mathcal{F}_n \subsetneq S$ is a multifilter on $\mathcal{M}_n$ iff the following condition holds, for each $j, k \leqslant n$, $j \neq k$, and $a, b \in S$:

- $a \cap_j b \in \mathcal{F}_n$ iff $a \in \mathcal{F}_n$ and $b \in \mathcal{F}_n$;

A multifilter $\mathcal{F}_n$ is a prime multifilter on $\mathcal{M}_n$ iff the following condition holds, for each $j, k \leqslant n$, $j \neq k$, and $a, b \in S$:

- $a \cup_j b \in \mathcal{F}_n$ iff $a \in \mathcal{F}_n$ or $b \in \mathcal{F}_n$.

Definition 161 (Logical multilattice)**.** [92, p. 319, Definition 2.3] A pair $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ is a *logical multilattice* iff $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ is a multilattice and $\mathcal{F}_n$ is a prime multifilter.

Definition 162 (Ultralogical multilattice)**.** [92, p. 319, Definition 2.4] A pair $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ is an *ultralogical multilattice* iff $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ is a logical multilattice and $\mathcal{U}_n$ satisfies the following condition, for each $j, k \leqslant n$, $j \neq k$, and $a \in S$:

- $a \in \mathcal{U}_n$ iff $-_k -_j a \notin \mathcal{U}_n$ ($\mathcal{U}_n$ is an ultramultifilter on $\mathcal{M}_n$).

Definition 163 (Language)**.** The formulas of the language $\mathscr{L}_M$ of the logic $\mathbf{ML}_n$ are built from the set $\mathcal{P} = \{p_n \mid n \in \mathbb{N}\}$ of propositional variables, negations $\neg_1, \ldots, \neg_n$, conjunctions $\wedge_1, \ldots, \wedge_n$, disjunctions $\vee_1, \ldots, \vee_n$, implications $\rightarrow_1, \ldots, \rightarrow_n$, and co-implications $\leftarrow_1, \ldots, \leftarrow_n$. The set of all $\mathscr{L}_M$-formulas $\mathscr{F}_M$ is defined in a standard inductive way. The language $\mathscr{L}_M^{\Box}$ of the logic $\mathbf{MML}_n^{\mathbf{S5}}$ extends $\mathscr{L}_M$ by modal operators of necessity $\Box_1, \ldots, \Box_n$ and possibility $\Diamond_1, \ldots, \Diamond_n$. The set of all $\mathscr{L}_M^{\Box}$-formulas $\mathscr{F}_M^{\Box}$ is defined in a standard inductive way.

Definition 164 (Valuation in $\mathbf{ML}_n$ and $\mathbf{MML}_n^{\mathbf{S5}}$)**.** A valuation v (of the logic $\mathbf{ML}_n$) is defined as a mapping from $\mathcal{P}$ to S. It is extended into complex formulas as follows: $v(\neg_j A) = -_j v(A)$, $v(A \wedge_j B) = v(A) \cap_j v(B)$, $v(A \vee_j B) = v(A) \cup_j v(B)$, $v(A \rightarrow_j B) = -_k -_j v(A) \cup_j v(B)$, and $v(A \leftarrow_j B) = v(A) \cap_j -_k -_j v(B)$. A modal valuation (of the logic $\mathbf{MML}_n^{\mathbf{S5}}$) has two additional cases: $v(\Box_j A) = I_j v(A)$, $v(\Diamond_j A) = C_j v(A)$.

Remark 165. In $\mathbf{ML}_n$, if $j, k \leqslant n$ and $j \neq k$, then $\neg_k \neg_j A$ is equivalent to $\neg_j \neg_k A$; $\neg_k \neg_j$ behaves as Boolean negation; A is equivalent to $\neg_j \neg_j A$.

Let us give a concrete example of a multilattice. Let $n = 2$. Then we obtain a bilattice $B = \langle S, \leqslant_1, \leqslant_2 \rangle$. Let $S = \{1, b, n, 0\}$ (the interpretation of these elements is the same as in **FDE**), let $\leqslant_1$ be such that $0 \leqslant_1 n \leqslant_1 1$ and $0 \leqslant_1 b \leqslant_1 1$ (n and b are incomparable), and let $\leqslant_2$ be such that $n \leqslant_2 1 \leqslant_2 b$ and $n \leqslant_1 0 \leqslant_1 b$ (1 and 0 are incomparable). The order $\leqslant_1$ is known as truth order, since it orders “degree of truth”; the order $\leqslant_2$ is known as knowledge or information order, since it orders “degree of information”; see e.g. [2, 55] for more details.

We will denote $\leqslant_1$ and $\leqslant_2$, respectively, as $\leqslant_t$ and $\leqslant_k$. The order $\leqslant_t$ produces involution, meet, and join operations which correspond to the connectives of **FDE**, denoted as $\neg, \wedge, \vee$; the order $\leqslant_k$ produces involution, meet, and join operations which due to Fitting [55] are known as conflation, consensus, and gullability, denoted as $-, \otimes, \oplus$ (see the truth tables below). One may easily observe that the set $\{1, b\}$ is a filter on B (it is also said to be the set of designated values).

A	$\neg$	$-$
1	0	1
b	b	n
n	n	b
0	1	0

$\wedge$	1	b	n	0
1	1	b	n	0
b	b	b	0	0
n	n	0	n	0
0	0	0	0	0

$\vee$	1	b	n	0
1	1	1	1	1
b	1	b	1	b
n	1	1	n	n
0	1	b	n	0

$\otimes$	1	b	n	0
1	1	1	n	n
b	1	b	n	0
n	n	n	n	n
0	n	0	n	0

$\oplus$	1	b	n	0
1	1	b	1	b
b	b	b	b	b
n	1	b	n	0
0	b	b	0	0

Definition 166 (Tarski multilattice)**.** A multilattice $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ is said to be a *Tarski* multilattice (or a modal multilattice with Tarski operators) iff for each $j \leqslant n$ the unary Tarski-style operations of *interior* I_j and *closure* C_j can be defined on S and satisfy the following conditions ($x, y \in S$):

$$I_j(\mathsf{1}_j) = \mathsf{1}_j; \qquad (\mathsf{1}_j \text{ is open})$$
$$C_j(\mathsf{0}_j) = \mathsf{0}_j; \qquad (\mathsf{0}_j \text{ is closed})$$
$$-_j I_j(x) = C_j(-_j x); \qquad (-_j I_j\text{-definition})$$
$$-_j C_j(x) = I_j(-_j x); \qquad (-_j C_j\text{-definition})$$
$$-_k I_j(x) = I_j(-_k x); \qquad (-_k I_j\text{-definition})$$
$$-_k C_j(x) = C_j(-_k x); \qquad (-_k C_j\text{-definition})$$
$$I_j(x) = -_j -_k C_j(-_j -_k x); \qquad (I\text{-definition})$$
$$C_j(x) = -_j -_k I_j(-_j -_k x); \qquad (C\text{-definition})$$
$$I_j(x) \leqslant_j x; \qquad (\text{decreasing})$$
$$I_j(x) = I_j I_j(x); \qquad (I\text{-idempotent})$$
$$I_j(x \cap_j y) \leqslant_j I_j(x) \cap_j I_j(y); \qquad (\text{sub-multiplicative})$$
$$x \leqslant_j C_j(x); \qquad (\text{increasing})$$
$$C_j(x) = C_j C_j(x); \qquad (C\text{-idempotent})$$
$$C_j(x) \cup_j C_j(y) \leqslant_j C_j(x \cup_j y). \qquad (\text{sub-additive})$$

Fact 167. Each Tarski multilattice satisfies the following conditions:

$$x \leqslant_j y \text{ implies } I_j(x) \leqslant_j I_j(y); \qquad (I\text{-monotonicity})$$
$$x \leqslant_j y \text{ implies } C_j(x) \leqslant_j C_j(y). \qquad (C\text{-monotonicity})$$

Proof. If $x \leqslant_j y$, then $I_j(x) = I_j(x \cap_j y) \leqslant_j I_j(x) \cap_j I_j(y) \leqslant_j I_j(y)$.
If $x \leqslant_j y$, then $C_j(x) \leqslant_j C_j(x) \cup_j C_j(y) \leqslant_j C_j(x \cup_j y) = C_j(y)$. □

Definition 168 (Kuratowski multilattice)**.** A Tarski multilattice $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ is said to be a *Kuratowski* one (or a modal multilattice with Kuratowski operators) iff for each $j \leqslant n$ the operations I_j and C_j satisfy the following conditions:

$$I_j(x \cap_j y) = I_j(x) \cap_j I_j(y); \qquad \text{(multiplicative)}$$
$$C_j(x) \cup_j C_j(y) = C_j(x \cup_j y). \qquad \text{(additive)}$$

Definition 169 (Halmos multilattice)**.** A Kuratowski multilattice $\mathcal{M}_n = \langle S, \leqslant_1, \ldots, \leqslant_n \rangle$ is said to be a *Halmos* one (or a modal multilattice with Halmos operators) iff for each $j \leqslant n$ the operations I_j and C_j satisfy the following conditions:

$$I_j(-_j I_j(x)) = -_j I_j(x); \qquad \text{(interior interconnection)}$$
$$C_j(-_j C_j(x)) = -_j C_j(x). \qquad \text{(closure interconnection)}$$

Definition 170. A pair $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ is a *Tarski (resp. Kuratowski, Halmos) ultralogical multilattice* iff $\mathcal{M}_n$ is a Tarski (resp. Kuratowski, Halmos) multilattice and $\mathcal{U}_n$ is an ultramultifilter on it.

Definition 171 (Entailment in multilattice logics)**.** For any finite sets of formulas Γ and Δ:

- $\Gamma \models_{\mathbf{MLL}_n} \Delta$ iff for each logical multilattice $\langle \mathcal{M}_n, \mathcal{F}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{U}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{U}_n$, for some $D \in \Delta$.
- $\Gamma \models_{\mathbf{ML}_n} \Delta$ iff for each *ultra*logical multilattice $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{U}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{U}_n$, for some $D \in \Delta$.
- $\Gamma \models_{\mathbf{MML}_n^{\mathbf{MNT4}}} \Delta$ iff for each Tarski ultralogical multilattice $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{U}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{U}_n$, for some $D \in \Delta$.
- $\Gamma \models_{\mathbf{MML}_n^{\mathbf{S4}}} \Delta$ iff for each Kuratowski ultralogical multilattice $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{U}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{U}_n$, for some $D \in \Delta$.
- $\Gamma \models_{\mathbf{MML}_n^{\mathbf{S5}}} \Delta$ iff for each Halmos ultralogical multilattice $\langle \mathcal{M}_n, \mathcal{U}_n \rangle$ and each valuation v, it holds that if $v(C) \in \mathcal{U}_n$, for each $C \in \Gamma$, then $v(D) \in \mathcal{U}_n$, for some $D \in \Delta$.

A sequent is called valid for a multilattice logic $\mathbf{L}$ iff $\Gamma \models_{\mathbf{L}} \Delta$ holds. When $\Gamma \Rightarrow \Delta$ is valid for $\mathbf{L}$, we write $\mathbf{L} \models \Gamma \Rightarrow \Delta$ or $\models_{\mathbf{L}} \Gamma \Rightarrow \Delta$.

Let us present a hypersequent calculus for $\mathbf{MML}_n^{\mathbf{S5}}$ from [71] (its non-modal part is a calculus for $\mathbf{ML}_n$; if we delete from the non-modal part rules for $\neg_k \neg_j$, implications, and coimplications, we obtain a calculus for $\mathbf{MLL}_n$ [68]; sequent calculi for $\mathbf{MML}_n^{\mathbf{S4}}$ and $\mathbf{MML}_n^{\mathbf{MNT4}}$ can be found in [69]). The axioms are as follows, for any propositional variable p[3]:

$$(Ax)\ p \Rightarrow p \qquad (Ax_\neg)\ \neg_j p \Rightarrow \neg_j p$$

The non-negated logical rules are as follows:

$$(\wedge_j \Rightarrow)\ \frac{A, B, \Gamma \Rightarrow \Delta \mid H}{A \wedge_j B, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \wedge_j)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \qquad \Gamma \Rightarrow \Delta, B \mid G}{\Gamma \Rightarrow \Delta, A \wedge_j B \mid H \mid G}$$

$$(\vee_j \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad B, \Gamma \Rightarrow \Delta \mid G}{A \vee_j B, \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow \vee_j)\ \frac{\Gamma \Rightarrow \Delta, A, B \mid H}{\Gamma \Rightarrow \Delta, A \vee_j B \mid H}$$

[3]By induction on the complexity of a formula it is possible to show that $A \Rightarrow A$ holds, for any $\mathscr{L}_M^{\Box}$-formula A.

$$(\to_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \qquad B, \Theta \Rightarrow \Lambda \mid G}{A \to_j B, \Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H \mid G} \qquad (\Rightarrow \to_j)\ \frac{A, \Gamma \Rightarrow \Delta, B \mid H}{\Gamma \Rightarrow \Delta, A \to_j B \mid H}$$

$$(\leftarrow_j \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta, B \mid H}{A \leftarrow_j B, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \leftarrow_j)\ \frac{\Gamma \Rightarrow \Delta, A \mid H \qquad B, \Theta \Rightarrow \Lambda \mid G}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, A \leftarrow_j B \mid H \mid G}$$

The jj-negated logical rules are as follows:

$$(\neg_j\wedge_j \Rightarrow)\ \frac{\neg_j A, \Gamma \Rightarrow \Delta \mid H \qquad \neg_j B, \Gamma \Rightarrow \Delta \mid G}{\neg_j(A \wedge_j B), \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow \neg_j\wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j A, \neg_j B \mid H}{\Gamma \Rightarrow \Delta, \neg_j(A \wedge_j B) \mid H}$$

$$(\neg_j\vee_j \Rightarrow)\ \frac{\neg_j A, \neg_j B, \Gamma \Rightarrow \Delta \mid H}{\neg_j(A \vee_j B), \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_j\vee_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j A \mid H \qquad \Gamma \Rightarrow \Delta, \neg_j B \mid G}{\Gamma \Rightarrow \Delta, \neg_j(A \vee_j B) \mid H \mid G}$$

$$(\neg_j\to_j \Rightarrow)\ \frac{\neg_j B, \Gamma \Rightarrow \Delta, \neg_j A \mid H}{\neg_j(A \to_j B), \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_j\to_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j B \mid H \qquad \neg_j A, \Theta \Rightarrow \Lambda \mid G}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_j(A \to_j B) \mid H \mid G}$$

$$(\neg_j\leftarrow_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_j B \mid H \qquad \neg_j A, \Theta \Rightarrow \Lambda \mid G}{\neg_j(A \leftarrow_j B), \Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H \mid G} \qquad (\Rightarrow \neg_j\leftarrow_j)\ \frac{\neg_j B, \Gamma \Rightarrow \Delta, \neg_j A \mid H}{\Gamma \Rightarrow \Delta, \neg_j(A \leftarrow_j B) \mid H}$$

$$(\neg_j\neg_j \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H}{\neg_j\neg_j A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_j\neg_j)\ \frac{\Gamma \Rightarrow \Delta, A \mid H}{\Gamma \Rightarrow \Delta, \neg_j\neg_j A \mid H}$$

The kj-negated logical rules as follows:

$$(\neg_k\wedge_j \Rightarrow)\ \frac{\neg_k A, \neg_k B, \Gamma \Rightarrow \Delta \mid H}{\neg_k(A \wedge_j B), \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_k\wedge_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k A \mid H \qquad \Gamma \Rightarrow \Delta, \neg_k B \mid G}{\Gamma \Rightarrow \Delta, \neg_k(A \wedge_j B) \mid H \mid G}$$

$$(\neg_k\vee_j \Rightarrow)\ \frac{\neg_k A, \Gamma \Rightarrow \Delta \mid H \qquad \neg_k B, \Gamma \Rightarrow \Delta \mid G}{\neg_k(A \vee_j B), \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow \neg_k\vee_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k A, \neg_k B \mid H}{\Gamma \Rightarrow \Delta, \neg_k(A \vee_j B) \mid H}$$

$$(\neg_k\to_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_k A \mid H \qquad \neg_k B, \Theta \Rightarrow \Lambda \mid G}{\neg_k(A \to_j B), \Gamma, \Theta \Rightarrow \Delta, \Lambda \mid H \mid G} \qquad (\Rightarrow \neg_k\to_j)\ \frac{\neg_k A, \Gamma \Rightarrow \Delta, \neg_k B \mid H}{\Gamma \Rightarrow \Delta, \neg_k(A \to_j B) \mid H}$$

$$(\neg_k\leftarrow_j \Rightarrow)\ \frac{\neg_k A, \Gamma \Rightarrow \Delta, \neg_k B \mid H}{\neg_k(A \leftarrow_j B), \Gamma \Rightarrow \Delta \mid H} \qquad (\neg_k\leftarrow_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_k A \mid H \qquad \neg_k B, \Theta \Rightarrow \Lambda \mid G}{\Gamma, \Theta \Rightarrow \Delta, \Lambda, \neg_k(A \leftarrow_j B) \mid H \mid G}$$

$$(\neg_k\neg_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid H}{\neg_k\neg_j A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_k\neg_j)\ \frac{A, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \neg_k\neg_j A \mid H}$$

The non-negated modal rules are as follows:

$$(\Box_j \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H}{\Box_j A \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \Box_j)\ \frac{\Rightarrow A \mid H}{\Rightarrow \Box_j A \mid H}$$

$$(\Diamond_j \Rightarrow)\ \frac{A \Rightarrow \mid H}{\Diamond_j A \Rightarrow \mid H} \qquad (\Rightarrow \Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, A \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \Diamond_j A \mid H}$$

The jj-negated modal rules are as follows:

$$(\neg_j\Box_j \Rightarrow)\ \frac{\neg_j A \Rightarrow \mid H}{\neg_j\Box_j A \Rightarrow \mid H} \qquad (\Rightarrow \neg_j\Box_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j A \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \neg_j\Box_j A \mid H}$$

$$(\neg_j\Diamond_j \Rightarrow)\ \frac{\neg_j A, \Gamma \Rightarrow \Delta \mid H}{\neg_j\Diamond_j A \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_j\Diamond_j)\ \frac{\Rightarrow \neg_j A \mid H}{\Rightarrow \neg_j\Diamond_j A \mid H}$$

The kj-negated modal rules are as follows:

$$(\neg_k\Box_j \Rightarrow)\ \frac{\neg_k A, \Gamma \Rightarrow \Delta \mid H}{\neg_k\Box_j A \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_k\Box_j)\ \frac{\Rightarrow \neg_k A \mid H}{\Rightarrow \neg_k\Box_j A \mid H}$$

$$(\neg_k\Diamond_j \Rightarrow)\ \frac{\neg_k A \Rightarrow \mid H}{\neg_k\Diamond_j A \Rightarrow \mid H} \qquad (\Rightarrow \neg_k\Diamond_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k A \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \neg_k\Diamond_j A \mid H}$$

For the case of non-standard modalities we propose the following rules. The non-negated rules for non-standard modalities:

$$(\rhd_j \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, A \mid G}{\rhd_j A \Rightarrow \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H \mid G} \qquad (\Rightarrow \rhd_j)\ \frac{\Rightarrow A \mid A \Rightarrow \mid H}{\Rightarrow\ \rhd_j A \mid H}$$

$$(\blacktriangleright_j \Rightarrow)\ \frac{\Rightarrow A \mid A \Rightarrow \mid H}{\blacktriangleright_j A \Rightarrow \mid H} \qquad (\Rightarrow \blacktriangleright_j)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, A \mid G}{\Rightarrow \blacktriangleright_j A \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H \mid G}$$

$$(\circ_j \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, A \mid G}{\circ_j A, \Theta \Rightarrow \Lambda \mid \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow \circ_j)\ \frac{\Rightarrow A \mid A, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \circ_j A \mid H}$$

$$(\bullet_j \Rightarrow)\ \frac{\Rightarrow A \mid A, \Gamma \Rightarrow \Delta \mid H}{\bullet_j A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \bullet_j)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, A \mid G}{\Theta \Rightarrow \Lambda, \bullet_j A \mid \Gamma \Rightarrow \Delta \mid H \mid G}$$

$$(\widetilde{\circ}_j \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, A \mid G}{\widetilde{\circ}_j A, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H \mid G} \qquad (\Rightarrow \widetilde{\circ}_j)\ \frac{\Gamma \Rightarrow \Delta, A \mid A \Rightarrow \mid H}{\Gamma \Rightarrow \Delta, \widetilde{\circ}_j A \mid H}$$

$$(\widetilde{\bullet}_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid A \Rightarrow \mid H}{\widetilde{\bullet}_j A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \widetilde{\bullet}_j)\ \frac{A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, A \mid G}{\Gamma \Rightarrow \Delta, \widetilde{\bullet}_j A \mid \Theta \Rightarrow \Lambda \mid H \mid G}$$

$$(\sim_j \Rightarrow)\ \frac{\Rightarrow A \mid H}{\sim_j A \Rightarrow \mid H} \qquad (\Rightarrow \sim_j)\ \frac{A, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \sim_j A \mid H}$$

$$(\dot{\sim}_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, A \mid H}{\dot{\sim}_j A \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \dot{\sim}_j)\ \frac{A \Rightarrow \mid H}{\Rightarrow \dot{\sim}_j A \mid H}$$

The jj-negated rules for non-standard modalities:

$$(\neg_j\rhd_j \Rightarrow)\ \frac{\Rightarrow \neg_j A \mid \neg_j A \Rightarrow \mid H}{\neg_j\rhd_j A \Rightarrow \mid H} \qquad (\Rightarrow \neg_j\rhd_j)\ \frac{\neg_j A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, \neg_j A \mid G}{\Rightarrow \neg_j\rhd_j A \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H \mid G}$$

$$(\neg_j\blacktriangleright_j \Rightarrow)\ \frac{\neg_j A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, \neg_j A \mid G}{\neg_j\blacktriangleright_j A \Rightarrow \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H \mid G} \qquad (\Rightarrow \neg_j\blacktriangleright_j)\ \frac{\Rightarrow \neg_j A \mid \neg_j A \Rightarrow \mid H}{\Rightarrow\ \neg_j\blacktriangleright_j A \mid H}$$

$$(\neg_j\circ_j \Rightarrow)\ \frac{\neg_j A \Rightarrow \mid \Gamma \Rightarrow \Delta, \neg_j A \mid H}{\neg_j\circ_j A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_j\circ_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j A \mid H \qquad \neg_j A, \Theta \Rightarrow \Lambda \mid G}{\Theta \Rightarrow \Lambda, \neg_j\circ A_j \mid \Gamma \Rightarrow \Delta \mid H \mid G}$$

$$(\neg_j\bullet_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_j A \mid H \qquad \neg_j A, \Theta \Rightarrow \Lambda \mid G}{\neg_j\bullet_j A, \Theta \Rightarrow \Lambda \mid \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow \neg_j\bullet_j)\ \frac{\neg_j A \Rightarrow \mid \Gamma \Rightarrow \Delta, \neg_j A \mid H}{\Gamma \Rightarrow \Delta, \neg_j\bullet_j A \mid H}$$

$$(\neg_j\widetilde{\circ}_j \Rightarrow)\ \frac{\neg_j A, \Gamma \Rightarrow \Delta \mid \Rightarrow \neg_j A \mid H}{\neg_j\widetilde{\circ}_j A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_j\widetilde{\circ}_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_j A \mid H \qquad \neg_j A, \Theta \Rightarrow \Lambda \mid G}{\Gamma \Rightarrow \Delta, \neg_j\widetilde{\circ}_j A \mid \Theta \Rightarrow \Lambda \mid H \mid G}$$

$$(\neg_j\widetilde{\bullet}_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_j A \mid H \qquad \neg_j A, \Theta \Rightarrow \Lambda \mid G}{\neg_j\widetilde{\bullet}_j A, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H \mid G} \qquad (\Rightarrow \neg_j\widetilde{\bullet}_j)\ \frac{\neg_j A, \Gamma \Rightarrow \Delta \mid \Rightarrow \neg_j A \mid H}{\Gamma \Rightarrow \Delta, \neg_j\widetilde{\bullet}_j A \mid H}$$

$$(\neg_j{\sim}_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_j A \mid H}{\neg_j{\sim}_j A \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_j{\sim}_j)\ \frac{\neg_j A \Rightarrow \mid H}{\Rightarrow \neg_j{\sim}_j A \mid H}$$

$$(\neg_j\dot{\sim}_j \Rightarrow)\ \frac{\Rightarrow \neg_j A \mid H}{\neg_j\dot{\sim}_j A \Rightarrow \mid H} \qquad (\Rightarrow \neg_j\dot{\sim}_j)\ \frac{\neg_j A, \Gamma \Rightarrow \Delta \mid H}{\Rightarrow \neg_j\dot{\sim}_j A \mid \Gamma \Rightarrow \Delta \mid H}$$

The kj-negated rules for non-standard modalities:

$$(\neg_k\rhd_j \Rightarrow)\ \frac{\neg_k A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, \neg_k A \mid G}{\neg_k\rhd_j A \Rightarrow \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H \mid G} \qquad (\Rightarrow \neg_k\rhd_j)\ \frac{\Rightarrow \neg_k A \mid \neg_k A \Rightarrow \mid H}{\Rightarrow\ \neg_k\rhd_j A \mid H}$$

$$(\neg_k\blacktriangleright_j \Rightarrow)\ \frac{\Rightarrow \neg_k A \mid \neg_k A \Rightarrow \mid H}{\neg_k\blacktriangleright_j A \Rightarrow \mid H} \qquad (\Rightarrow \neg_k\blacktriangleright_j)\ \frac{\neg_k A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, \neg_k A \mid G}{\Rightarrow \neg_k\blacktriangleright_j A \mid \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H \mid G}$$

$$(\neg_k\circ_j \Rightarrow)\ \frac{\neg_k A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, \neg_k A \mid G}{\neg_k\circ_j A, \Theta \Rightarrow \Lambda \mid \Gamma \Rightarrow \Delta \mid H \mid G} \qquad (\Rightarrow \circ_j)\ \frac{\Rightarrow \neg_k A \mid \neg_k A, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta, \neg_k\circ_j A \mid H}$$

$$(\neg_k\bullet_j \Rightarrow)\ \frac{\Rightarrow \neg_k A \mid \neg_k A, \Gamma \Rightarrow \Delta \mid H}{\neg_k\bullet_j A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_k\bullet_j)\ \frac{\neg_k A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, \neg_k A \mid G}{\Theta \Rightarrow \Lambda, \neg_k\bullet_j A \mid \Gamma \Rightarrow \Delta \mid H \mid G}$$

$$(\neg_k\widetilde{\circ}_j \Rightarrow)\ \frac{\neg_k A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, \neg_k A \mid G}{\neg_k\widetilde{\circ}_j A, \Gamma \Rightarrow \Delta \mid \Theta \Rightarrow \Lambda \mid H \mid G} \qquad (\Rightarrow \neg_k\widetilde{\circ}_j)\ \frac{\Gamma \Rightarrow \Delta, \neg_k A \mid \neg_k A \Rightarrow \mid H}{\Gamma \Rightarrow \Delta, \neg_k\widetilde{\circ}_j A \mid H}$$

$$(\neg_k\widetilde{\bullet}_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_k A \mid \neg_k A \Rightarrow \mid H}{\neg_k\widetilde{\bullet}_j A, \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_k\widetilde{\bullet}_j)\ \frac{\neg_k A, \Gamma \Rightarrow \Delta \mid H \qquad \Theta \Rightarrow \Lambda, \neg_k A \mid G}{\Gamma \Rightarrow \Delta, \neg_k\widetilde{\bullet}_j A \mid \Theta \Rightarrow \Lambda \mid H \mid G}$$

$$(\neg_k{\sim}_j \Rightarrow)\ \frac{\Rightarrow \neg_k A \mid H}{\neg_k{\sim}_j A \Rightarrow \mid H} \qquad (\Rightarrow \neg_k{\sim}_j)\ \frac{\neg_k A, \Gamma \Rightarrow \Delta \mid H}{\Gamma \Rightarrow \Delta \mid \Rightarrow \neg_k{\sim}_j A \mid H}$$

$$(\neg_k\dot{\sim}_j \Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \neg_k A \mid H}{\neg_k\dot{\sim}_j A \Rightarrow \mid \Gamma \Rightarrow \Delta \mid H} \qquad (\Rightarrow \neg_k\dot{\sim}_j)\ \frac{\neg_k A \Rightarrow \mid H}{\Rightarrow \neg_k\dot{\sim}_j A \mid H}$$

The notion of a proof in this hypersequent calculus is defined in a standard way.

The interdefinability laws $\Box_j A \Leftrightarrow \neg_j\neg_k\Diamond_j\neg_j\neg_k A$ and $\Diamond_j A \Leftrightarrow \neg_j\neg_k\Box_j\neg_j\neg_k A$ are provable in $\mathbf{MML}_n^{\mathbf{S5}}$. Besides, sequents $\neg_j\Diamond_j A \Leftrightarrow \Box_j\neg_j A$, $\neg_j\Box_j A \Leftrightarrow \Diamond_j\neg_j A$, $\neg_k\Box_j A \Leftrightarrow \Box_j\neg_k A$, and $\neg_k\Diamond_j A \Leftrightarrow \Diamond_j\neg_k A$ are provable in $\mathbf{MML}_n^{\mathbf{S5}}$ as well.

Let us now present Kripke semantics for $\mathbf{MML}_n^{\mathbf{S5}}$ developed on the basis of Kripke semantics $\mathbf{MML}_n$ from [92] (it is a bit simplified version of the semantics for $\mathbf{MML}_n^{\mathbf{S5}}$ from [71]). We write $\mathcal{P} \cup {}^{\neg}\mathcal{P}$ for the set of propositional variables joint with the set of negated propositional variables; to be more precise, ${}^{\neg}\mathcal{P} = \{\neg_j p \mid p \in \mathcal{P}, j \leqslant n\}$.

Definition 172. A triple $\langle W, R, \models^{\mathfrak{p}} \rangle$ is a $\mathbf{MML}_n^{\mathbf{S5}}$-model iff $W \neq \emptyset$, $R = W \times W$, and a paradefinite valuation $\models^{\mathfrak{p}}$ is a mapping $\models^{\mathfrak{p}} \colon \mathcal{P} \cup {}^{\neg}\mathcal{P} \mapsto 2^W$ from the set of propositional variables and negated propositional variable to the power-set of W. We write $w \models^{\mathfrak{p}} p$ for $w \in \models^{\mathfrak{p}}(p)$, where $w \in W$. The paradefinite valuation $\models^{\mathfrak{p}}$ is extended to the mapping from the set of all $\mathscr{L}_M$-formulas to 2^W as follows, $x \in W$:

(1) $x \models^{\mathfrak{p}} A \wedge_j B$ iff $x \models^{\mathfrak{p}} A$ and $x \models^{\mathfrak{p}} B$,

(2) $x \models^{\mathfrak{p}} A \vee_j B$ iff $x \models^{\mathfrak{p}} A$ or $x \models^{\mathfrak{p}} B$,

(3) $x \models^{\mathfrak{p}} A \rightarrow_j B$ iff $w \not\models^{\mathfrak{p}} A$ or $x \models^{\mathfrak{p}} B$,

(4) $x \models^{\mathfrak{p}} A \leftarrow_j B$ iff $x \models^{\mathfrak{p}} A$ and $x \not\models^{\mathfrak{p}} B$,

(5) $x \models^{\mathfrak{p}} \Box_j A$ iff $\forall_{y \in W}(R(x,y)$ implies $y \models^{\mathfrak{p}} A)$,

(6) $x \models^{\mathfrak{p}} \Diamond_j A$ iff $\exists_{y \in W}(R(x,y)$ and $y \models^{\mathfrak{p}} A)$,

(7) $x \models^{\mathfrak{p}} \neg_j(A \wedge_j B)$ iff $x \models^{\mathfrak{p}} \neg_j A$ or $x \models^{\mathfrak{p}} \neg_j B$,

(8) $x \models^{\mathfrak{p}} \neg_j(A \vee_j B)$ iff $x \models^{\mathfrak{p}} \neg_j A$ and $x \models^{\mathfrak{p}} \neg_j B$,

(9) $x \models^{\mathfrak{p}} \neg_j(A \rightarrow_j B)$ iff $x \models^{\mathfrak{p}} \neg_j B$ and $x \not\models^{\mathfrak{p}} \neg_j A$,

(10) $x \models^{\mathfrak{p}} \neg_j(A \leftarrow_j B)$ iff $x \not\models^{\mathfrak{p}} \neg_j B$ or $x \models^{\mathfrak{p}} \neg_j A$,

(11) $x \models^{\mathfrak{p}} \neg_j\neg_j A$ iff $x \models^{\mathfrak{p}} A$,

(12) $x \models^{\mathfrak{p}} \neg_j\Box_j A$ iff $\exists_{y\in W}(R(x,y)$ and $y \models^{\mathfrak{p}} \neg_j A)$,

(13) $x \models^{\mathfrak{p}} \neg_j\Diamond_j A$ iff $\forall_{y\in W}(R(x,y)$ implies $y \models^{\mathfrak{p}} \neg_j A)$,

(14) $x \models^{\mathfrak{p}} \neg_k(A \wedge_j B)$ iff $x \models^{\mathfrak{p}} \neg_k A$ and $x \models^{\mathfrak{p}} \neg_k B$,

(15) $x \models^{\mathfrak{p}} \neg_k(A \vee_j B)$ iff $x \models^{\mathfrak{p}} \neg_k A$ or $x \models^{\mathfrak{p}} \neg_k B$,

(16) $x \models^{\mathfrak{p}} \neg_k(A \rightarrow_j B)$ iff $x \not\models^{\mathfrak{p}} \neg_k A$ or $x \models^{\mathfrak{p}} \neg_k B$,

(17) $x \models^{\mathfrak{p}} \neg_k(A \leftarrow_j B)$ iff $x \models^{\mathfrak{p}} \neg_k A$ and $x \not\models^{\mathfrak{p}} \neg_k B$,

(18) $x \models^{\mathfrak{p}} \neg_k\neg_j A$ iff $x \not\models^{\mathfrak{p}} A$,

(19) $x \models^{\mathfrak{p}} \neg_k\Box_j A$ iff $\forall_{y\in W}(R(x,y)$ implies $y \models^{\mathfrak{p}} \neg_k A)$,

(20) $x \models^{\mathfrak{p}} \neg_k\Diamond_j A$ iff $\exists_{y\in W}(R(x,y)$ and $y \models^{\mathfrak{p}} \neg_k A)$.

If we consider non-standard modalities as well, then the following clauses are appropriate:

(21) $x \models^{\mathfrak{p}} \rhd_j A$ iff $\forall_{y\in W}(R(x,y)$ implies $y \models^{\mathfrak{p}} A)$ or $\forall_{y\in W}(R(x,y)$ implies $y \not\models^{\mathfrak{p}} A)$,

(22) $x \models^{\mathfrak{p}} \neg_j\rhd_j A$ iff $\exists_{y\in W}(R(x,y)$ and $y \models^{\mathfrak{p}} \neg_j A)$ and $\exists_{y\in W}(R(x,y)$ and $y \not\models^{\mathfrak{p}} \neg_j A)$,

(23) $x \models^{\mathfrak{p}} \neg_k\rhd_j A$ iff $\forall_{y\in W}(R(x,y)$ implies $y \models^{\mathfrak{p}} \neg_k A)$ or $\forall_{y\in W}(R(x,y)$ implies $y \not\models^{\mathfrak{p}} \neg_k A)$,

(24) $x \models^{\mathfrak{p}} \blacktriangleright_j A$ iff $\exists_{y\in W}(R(x,y)$ and $y \models^{\mathfrak{p}} A)$ and $\exists_{y\in W}(R(x,y)$ and $y \not\models^{\mathfrak{p}} A)$,

(25) $x \models^{\mathfrak{p}} \neg_j\blacktriangleright_j A$ iff $\forall_{y\in W}(R(x,y)$ implies $y \models^{\mathfrak{p}} \neg_j A)$ or $\forall_{y\in W}(R(x,y)$ implies $y \not\models^{\mathfrak{p}} \neg_j A)$,

(26) $x \models^{\mathfrak{p}} \neg_k\blacktriangleright_j A$ iff $\exists_{y\in W}(R(x,y)$ and $y \models^{\mathfrak{p}} \neg_k A)$ and $\exists_{y\in W}(R(x,y)$ and $y \not\models^{\mathfrak{p}} \neg_k A)$,

(27) $x \models^{\mathfrak{p}} \circ_j A$ iff $x \not\models^{\mathfrak{p}} A$ or $\forall_{y\in W}(R(x,y)$ implies $y \models^{\mathfrak{p}} A)$,

(28) $x \models^{\mathfrak{p}} \neg_j\circ_j A$ iff $x \models^{\mathfrak{p}} \neg_j A$ and $\exists_{y\in W}(R(x,y)$ and $y \not\models^{\mathfrak{p}} \neg_j A)$,

(29) $x \models^{\mathfrak{p}} \neg_k\circ_j A$ iff $x \not\models^{\mathfrak{p}} \neg_k A$ or $\forall_{y\in W}(R(x,y)$ implies $y \models^{\mathfrak{p}} \neg_k A)$,

(30) $x \models^{\mathfrak{p}} \bullet_j A$ iff $x \models^{\mathfrak{p}} A$ and $\exists_{y\in W}(R(x,y)$ and $y \not\models^{\mathfrak{p}} A)$,

(31) $x \models^{\mathfrak{p}} \neg_j\bullet_j A$ iff $x \not\models^{\mathfrak{p}} \neg_j A$ or $\forall_{y\in W}(R(x,y)$ implies $y \models^{\mathfrak{p}} \neg_j A)$,

(32) $x \models^{\mathfrak{p}} \neg_k\bullet_j A$ iff $x \models^{\mathfrak{p}} \neg_k A$ and $\exists_{y\in W}(R(x,y)$ and $y \not\models^{\mathfrak{p}} \neg_k A)$,

(33) $x \models^{\mathfrak{p}} \widetilde{\circ}_j A$ iff $x \models^{\mathfrak{p}} A$ or $\forall_{y\in W}(R(x,y)$ implies $y \not\models^{\mathfrak{p}} A)$,

(34) $x \models^{\mathfrak{p}} \neg_j\widetilde{\circ}_j A$ iff $x \not\models^{\mathfrak{p}} \neg_j A$ and $\exists_{y\in W}(R(x,y)$ and $y \models^{\mathfrak{p}} \neg_j A)$,

(35) $x \models^{\mathfrak{p}} \neg_k\widetilde{\circ}_j A$ iff $x \models^{\mathfrak{p}} \neg_k A$ or $\forall_{y\in W}(R(x,y)$ implies $y \not\models^{\mathfrak{p}} \neg_k A)$,

(36) $x \models^{\mathfrak{p}} \widetilde{\bullet}_j A$ iff $x \not\models^{\mathfrak{p}} A$ and $\exists_{y\in W}(R(x,y)$ and $y \models^{\mathfrak{p}} A)$,

(37) $x \models^{\mathfrak{p}} \neg_j\widetilde{\bullet}_j A$ iff $x \models^{\mathfrak{p}} \neg_j A$ or $\forall_{y\in W}(R(x,y)$ implies $y \not\models^{\mathfrak{p}} \neg_j A)$,

(38) $x \models^{\mathfrak{p}} \neg_k\widetilde{\bullet}_j A$ iff $x \not\models^{\mathfrak{p}} \neg_k A$ and $\exists_{y\in W}(R(x,y)$ and $y \models^{\mathfrak{p}} \neg_k A)$,

(39) $x \models^{\mathfrak{p}} \sim_j A$ iff $\exists_{y\in W}(R(x,y)$ and $y \not\models^{\mathfrak{p}} A)$,

(40) $x \models^{\mathfrak{p}} \neg_j\sim_j A$ iff $\forall_{y\in W}(R(x,y)$ implies $y \models^{\mathfrak{p}} A)$,

(41) $x \models^{\mathfrak{p}} \neg_k\sim_j A$ iff $\exists_{y\in W}(R(x,y)$ and $y \not\models^{\mathfrak{p}} \neg_k A)$.

An $\mathscr{L}_M$-formula A is true in a $\mathbf{MML}_n^{\mathbf{S5}}$-model $\langle W, R, \models^{\mathfrak{p}}\rangle$ iff $x \models^{\mathfrak{p}} A$ for any $x \in W$, and is $\mathbf{MML}_n^{\mathbf{S5}}$-valid in a frame $\langle W, R\rangle$ iff it is true for every paradefinite valuation $\models^{\mathfrak{p}}$ on this frame.

Obviously, we can omit the relation R in this conditions, since $R = W \times W$; e.g.:

- $x \models^{\mathfrak{p}} \neg_k\Box_j A$ iff $\forall_{y\in W}\ y \models^{\mathfrak{p}} \neg_k A$,
- $x \models^{\mathfrak{p}} \neg_k\Diamond_j A$ iff $\exists_{y\in W}\ y \models^{\mathfrak{p}} \neg_k A$.

If in the above described model $\langle W, R, \models^{\mathfrak{p}}\rangle$ there is no restriction on R, then it is a $\mathbf{MML}_n^{\mathbf{K}}$-model; if R is reflexive and transitive, then it is a $\mathbf{MML}_n^{\mathbf{S4}}$-model, and so on. We can obtain modal multilattice logics by changing the properties of R in the same way as we can obtain modal logics based on classical propositional logic.

Theorem 173 (Strong soundness)**.** *Let* $\mathbf{L}$ *be* $\mathbf{MML}_n^{\mathbf{S5}}$ *or any of its extensions by non-standard modalities or a logic obtained from* $\mathbf{MML}_n^{\mathbf{S5}}$ *by the replacement of* $\Box_j$ *and* $\Diamond_j$ *with non-standard modalities. For each finite set of hypersequents* $\mathscr{H} \cup \{H\}$*, if* $\mathscr{H} \vdash_{\mathbb{HS}\mathbf{L}} H$*, then* $\mathscr{H} \models_{\mathbf{L}} H$*.*

Proof. Similarly to Theorem 5. □

Theorem 174 (Strong completeness)**.** *Let* $\mathbf{L}$ *be* $\mathbf{MML}_n^{\mathbf{S5}}$ *or any of its extensions by non-standard modalities or a logic obtained from* $\mathbf{MML}_n^{\mathbf{S5}}$ *by the replacement of* $\Box_j$ *and* $\Diamond_j$ *with non-standard modalities. For each finite set of hypersequents* $\mathscr{H} \cup \{H\}$*, if* $\mathscr{H} \models_{\mathbf{L}} H$*, then* $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H$*.*

Proof. Similarly to Theorem 6. We use the notions and notation from the proof of Theorem 6. Instead of writing $\langle W, R, \vDash^{\mathfrak{p}}\rangle$ we write just $\langle W, \vDash^{\mathfrak{p}}\rangle$, since $R = W \times W$.

We need to change a bit the definition of a valuation. Let $\vDash^{\mathfrak{p}}$ be the valuation such that $w \vDash^{\mathfrak{p}} p$ iff $p \in \Gamma_w$ and $w \vDash^{\mathfrak{p}} \neg_j p$ iff $\neg_j p \in \Gamma_w$, for each $p \in \mathcal{P}$. We need to prove that for each $A \in \mathbb{F}$ and each maximal component w of H^* it holds that:

(a) $A \in \Gamma_w$ implies $w \vDash^{\mathfrak{p}} A$,

(b) $A \in \Delta_w$ implies $w \nvDash^{\mathfrak{p}} A$.

The proof is by induction on the complexity of A. The basic case follows from the definition of $\vDash^{\mathfrak{p}}$.

Let A be $\neg_k(B \wedge_j C)$. Assume that $\neg_k(B \wedge_j C) \in \Gamma_w$. Suppose that $\neg_k B \notin \Gamma_w$ or $\neg_k C \notin \Gamma_w$. By the maximality of w, the sequent $\neg_k B, \neg_k C, \Gamma_w \Rightarrow \Delta_w$ is not a component of H^*. Since H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid \neg_k B, \neg_k C, \Gamma_w \Rightarrow \Delta_w$. By the rule $(\neg_k\wedge_j \Rightarrow)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid \neg_k(B \wedge_j C), \Gamma_w \Rightarrow \Delta_w$. By (IW⇒), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid \Gamma_w \Rightarrow \Delta_w$, i.e., $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid w$. By (EC), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^*$. Contradiction. Hence, $\neg_k B \in \Gamma_w$ and $\neg_k C \in \Gamma_w$. By the induction hypothesis, $w \vDash^{\mathfrak{p}} \neg_k B$ and $w \vDash^{\mathfrak{p}} \neg_k C$. Thus, $w \vDash^{\mathfrak{p}} \neg_k(B \wedge_j C)$.

Assume that $\neg_k(B \wedge_j C) \in \Delta_w$. Suppose that $\neg_k B \notin \Delta_w$ and $\neg_k C \notin \Delta_w$. By the maximality of w and the fact that H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \neg_k B$ and $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \neg_k C$. By the rule $(\Rightarrow \neg_k\wedge_j)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid \Gamma_w \Rightarrow \Delta_w, \neg_k(B \wedge_j C)$. By (IW⇒) and (EC), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^*$. Contradiction. Hence, $\neg_k B \in \Delta_w$ or $\neg_k C \in \Delta_w$. By the induction hypothesis, $w \nvDash^{\mathfrak{p}} \neg_k B$ or $w \nvDash^{\mathfrak{p}} \neg_k C$. Thus, $w \nvDash^{\mathfrak{p}} \neg_k(B \wedge_j C)$.

Let A be $\neg_j\neg_j B$. Suppose that $\neg_j\neg_j B \in \Gamma_w$. Assume that $B \notin \Gamma_w$. By the maximality of w and the fact that H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid B, \Gamma_w \Rightarrow \Delta_w$. By the rule $(\neg_j\neg_j \Rightarrow)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid \neg_j\neg_j B, \Gamma_w \Rightarrow \Delta_w$. By (IW⇒) and (EC), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^*$. Contradiction. Hence, $B \in \Gamma_w$. By the induction hypothesis, $w \vDash^{\mathfrak{p}} B$. Thus, $w \vDash^{\mathfrak{p}} \neg_j\neg_j B$.

Suppose that $\neg_j\neg_j B \in \Delta_w$. Assume that $B \notin \Delta_w$. Similarly to the previous case, use the rule $(\Rightarrow \neg_j\neg_j)$ to obtain a contradiction. Then $B \in \Delta_w$ and $w \nvDash^{\mathfrak{p}} B$. Thus, $w \nvDash^{\mathfrak{p}} \neg_j\neg_k B$.

Let A be $\neg_j\neg_k B$. Suppose that $\neg_j\neg_k B \in \Gamma_w$. Assume that $B \notin \Delta_w$. By the maximality of w and the fact that H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid \Gamma_w \Rightarrow \Delta_w, B$. By the rule $(\neg_j\neg_k \Rightarrow)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid \neg_j\neg_k B, \Gamma_w \Rightarrow \Delta_w$. By (IW⇒) and (EC), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^*$. Contradiction. Hence, $B \in \Delta_w$. By the induction hypothesis, $w \nvDash^{\mathfrak{p}} B$. Thus, $w \vDash^{\mathfrak{p}} \neg_j\neg_k B$.

Suppose that $\neg_j\neg_k B \in \Delta_w$. Assume that $B \notin \Gamma_w$. Similarly to the previous case, use the rule $(\Rightarrow \neg_j\neg_k)$ to obtain a contradiction. Hence, $B \in \Gamma_w$, then $w \vDash^{\mathfrak{p}} B$. Thus, $w \nvDash^{\mathfrak{p}} \neg_j\neg_k B$.

Let A be $\neg_j\Diamond_j B$. Suppose that $\neg_j\Diamond_j B \in \Gamma_w$. Assume that there is $y \in W$ such that $\neg_j B \notin \Gamma_y$. Since y is maximal, $\neg_j B, \Gamma_y \Rightarrow \Delta_y \notin H^*$. Since H^* is an $\mathbb{F}$-hypersequent, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid \neg_j B, \Gamma_y \Rightarrow \Delta_y$. By the rule $(\neg_j\Diamond_j \Rightarrow)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid \neg_j\Diamond_j B \Rightarrow \mid \Gamma_y \Rightarrow \Delta_y$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid A \Rightarrow \mid y$. By the rule (Merge), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid A \Rightarrow$. Since $A \in \Gamma_w$, by (IW⇒), (⇒IW), and (Merge), we get $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^* \mid w$. By (Merge), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HS}\mathbf{L}} H^*$. Contradiction. Hence, for each $x \in W$, $\neg_j B \in \Gamma_x$. By the induction hypothesis for B, for each $x \in W$, $x \vDash_{\mathfrak{p}} \neg_j B$. Thus, $w \vDash_{\mathfrak{p}} A$.

Suppose that $\neg_j\Diamond_j B \in \Delta_w$. Assume that $\Rightarrow \neg_j B \notin H^*$. Then $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow \neg_j B$, since H^* is an $\mathbb{F}$-sequence. By $(\Rightarrow \neg_j\Diamond_j)$, $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow \neg_j\Diamond_j B$, i.e., $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid \Rightarrow A$. Since $A \in \Delta_w$, by (EC) and (Merge), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^* \mid w$. By (Merge), $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H^*$. Contradiction. Hence, $\Rightarrow \neg_j B \in H^*$. Therefore, there is a $y \in W$ such that $\neg_j B \in \Delta_y$. By the induction hypothesis for B, there is a $y \in W$ such that $y \nvDash_{\mathfrak{p}} \neg_j B$. Therefore, $w \nvDash_{\mathfrak{p}} A$.

The other cases are similar to the previous ones.

The last step of the proof is to show that $\langle W, \vDash^{\mathfrak{p}}\rangle$ is a model for $\mathscr{H}$, but not for H. Analogous to Theorem 6. □

Corollary 175. *Let* $\mathbf{L}$ *be* $\mathbf{MML}_n^{\mathbf{S5}}$ *or any of its extensions by non-standard modalities or a logic obtained from* $\mathbf{MML}_n^{\mathbf{S5}}$ *by the replacement of* $\Box_j$ *and* $\Diamond_j$ *with non-standard modalities. For each finite set of hypersequents* $\mathscr{H} \cup \{H\}$, $\mathscr{H} \vdash_{\mathbb{HSL}} H$ *iff* $\mathscr{H} \models_{\mathbf{L}} H$.

Proof. Follows from Theorems 173 and 174. □

Corollary 176. *Let* $\mathbf{L}$ *be* $\mathbf{MML}_n^{\mathbf{S5}}$ *or any of its extensions by non-standard modalities or a logic obtained from* $\mathbf{MML}_n^{\mathbf{S5}}$ *by the replacement of* $\Box_j$ *and* $\Diamond_j$ *with non-standard modalities. Let* $\mathscr{H} \cup \{H\}$ *be a finite set of hypersequents. Then* $\mathscr{H} \vdash_{\mathbb{HSL}} H$ *implies* $\mathscr{H} \vdash^{\mathrm{cf}}_{\mathbb{HSL}} H$.

Proof. Follows from Theorem 6. Notice that in the proof of this theorem, (Cut) is used only once in order to show that $\langle W, \vartheta\rangle$ is a model for $\mathscr{H}$ and is applied only to formulas which belong to $\mathscr{H}$. □

Corollary 177 (Cut admissibility)**.** *Let* $\mathbf{L}$ *be* $\mathbf{MML}_n^{\mathbf{S5}}$ *or any of its extensions by non-standard modalities or a logic obtained from* $\mathbf{MML}_n^{\mathbf{S5}}$ *by the replacement of* $\Box_j$ *and* $\Diamond_j$ *with non-standard modalities. Let* H *be a hypersequent. Then* $\vdash_{\mathbb{HSL}} H$ *implies that there is a cut-free proof of* H *in* $\mathbb{HSL}$.

Proof. Put $\mathscr{H} = \emptyset$ in the proof of Theorem 174. Then the only application of (Cut) in the proof of this Theorem disappears. □

Corollary 178 (Negated subformula property)**.** *Let* $\mathbf{L}$ *be* $\mathbf{MML}_n^{\mathbf{S5}}$ *or any of its extensions by non-standard modalities or a logic obtained from* $\mathbf{MML}_n^{\mathbf{S5}}$ *by the replacement of* $\Box_j$ *and* $\Diamond_j$ *with non-standard modalities. For every hypersequent which is provable in* $\mathbb{HSL}$, *there is a proof such that each formula which occurs in it is either a subformula or a* j*-negation of the formulas which occur in the conclusion.*

Proof. Follows from Corollary 177 and the fact that in any of the rules of $\mathbb{HSL}$ each formula which occurs in the premises is either a subformula or a j-negation of the formulas which occur in the conclusion. □

Theorem 179 (Constructive elimination of cuts)**.** *Let* $\mathbf{L}$ *be* $\mathbf{MML}_n^{\mathbf{S5}}$ *or any of its extensions by non-standard modalities or a logic obtained from* $\mathbf{MML}_n^{\mathbf{S5}}$ *by the replacement of* $\Box_j$ *and* $\Diamond_j$ *with non-standard modalities. If a derivation* $\mathfrak{D}$ *in* $\mathbb{HSL}$ *has an application of* (Cut), *then it can be transformed into a cut-free derivation* $\mathfrak{D}'$.

Proof. By the same method as Theorem 13. The complexity $\mathfrak{c}(A)$ of an $\mathscr{L}_M$-formula A is defined as follows:

- $\mathfrak{c}(p) = \mathfrak{c}(\neg_j p)$, for any $p \in \mathcal{P}$,
- if $A \neq p$ and $A \neq \neg_j p$, then $\mathfrak{c}(\neg_j A) = \mathfrak{c}(A) + 1$,
- $\mathfrak{c}(M_j A) = \mathfrak{c}(A) + 2$, where M_j is any modal operator among the considered for $\mathbf{MML}_n$,
- $\mathfrak{c}(A \star_j B) = \mathfrak{c}(A) + \mathfrak{c}(B) + 2$, where $\star_j$ is any binary connective of $\mathbf{MML}_n$.

Other notions from the proof of Theorem 13 do not require changes. Right and left reduction lemmas have to be proved. The proofs are analogous to proofs of Lemmas 11 and 12. As an example, we consider a case needed for a proof of the right reduction lemma.

The rule of the last inference of $\mathfrak{D}_2$ is $(\neg_j\Diamond_j \Rightarrow)$.

Subcase 1. A is principal in $\mathfrak{D}_2$ and $A = \neg_j\Diamond_j B$. The last step of $\mathfrak{D}_2$ is as follows:

$$\frac{\neg_j B, \neg_j\Diamond_j B^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \neg_j\Diamond_j B^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid G}{\neg_j\Diamond_j B \Rightarrow \mid \neg_j\Diamond_j B^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \neg_j\Diamond_j B^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid G}$$

The last step of $\mathfrak{D}_1$ is as follows:

$$\frac{\Rightarrow \neg_j B \mid H}{\Rightarrow \neg_j\Diamond_j B \mid H}$$

We should obtain

$$\Rightarrow \mid \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G$$

By the induction hypothesis, we obtain a derivation $\mathfrak{D}_3$ of the following hypersequent such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$:

$$\neg_j B, \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_2^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G.$$

Using this hypersequent and $\Rightarrow \neg_j B \mid H$, by (Cut) and other structural rules, we get

$$\cfrac{\cfrac{\cfrac{\Rightarrow \neg_j B \mid H \qquad \neg_j B, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G}{\mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid H \mid G}}{\mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G}}{\Rightarrow \mid \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G}$$

Subcase 2. The rule of the last inference of $\mathfrak{D}_2$ is $(\neg_j\Diamond_j \Rightarrow)$ and the principal formula in $\mathfrak{D}_2$ is not A. The last step of $\mathfrak{D}_2$ is as follows:

$$\frac{\neg_j B, A^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid G}{\neg_j\Diamond_j B \Rightarrow \mid A^{i_1}, \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid A^{i_n}, \mathfrak{A}_n^{\Theta\Lambda} \mid G}$$

The last step of $\mathfrak{D}_1$ is as follows: $\Gamma \Rightarrow \Delta, A \mid H$.

We should obtain

$$\neg_j\Diamond_j B \Rightarrow \mid \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G$$

By the induction hypothesis, we obtain a derivation $\mathfrak{D}_3$ of the following hypersequent such that $\mathfrak{r}(\mathfrak{D}_3) \leq \mathfrak{c}(A)$:

$$\neg_j B, \mathfrak{A}_{i_1}^{\Gamma\Delta} \times \mathfrak{A}_1^{\Theta\Lambda} \mid \mathfrak{A}_{i_2}^{\Gamma\Delta} \times \mathfrak{A}_2^{\Theta\Lambda} \mid \ldots \mid \mathfrak{A}_{i_n}^{\Gamma\Delta} \times \mathfrak{A}_n^{\Theta\Lambda} \mid H \mid G.$$

Applying $(\neg_j\Diamond_j \Rightarrow)$, we get the required result. □

We can extend our approach to other modal multilattice logics: $\mathbf{MML}_n^{\mathbf{K}}$ and its reflexive, serial, transitive, and symmentic extensions. We are not going to offer algebraic semantics for them (although an algebraic semantics for $\mathbf{MML}_n^{\mathbf{S4}}$ has already been mentioned), but have briefly noted how Kripke semantics can be obtained for them. Now we would like to formulate nested sequent calculi for these logics. First, we need to formulate a nested sequent calculus for $\mathbf{MML}_n^{\mathbf{K}}$. Second, to obtain nested sequent calculi for its reflexive, serial, transitive, and symmentic extensions, we need to add the corresponding special structural rules described in Chapter 1. The nested sequent calculus for $\mathbf{MML}_n^{\mathbf{K}}$ contains all the axioms and rules (in a nested sequent formulation) for $\mathbf{ML}_n$ as well as the following modal rules. The non-negated modal rules:

$$[\Box_j \Rightarrow]\ \frac{\mathfrak{N}[\Box_j A, \Gamma \Rightarrow \Delta/(A, \Theta \Rightarrow \Lambda/X)]}{\mathfrak{N}[\Box_j A, \Gamma \Rightarrow \Delta/(\Theta \Rightarrow \Lambda/X)]} \qquad [\Rightarrow \Box_j]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/ \Rightarrow A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \Box_j A]}$$

$$[\Diamond_j \Rightarrow]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / A \Rightarrow]}{\mathfrak{N}[\Diamond_j A, \Gamma \Rightarrow \Delta]} \qquad [\Rightarrow \Diamond_j]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \Diamond_j A/(\Theta \Rightarrow \Lambda, A/X)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \Diamond_j A/(\Theta \Rightarrow \Lambda/X)]}$$

The jj-negated modal rules:

$$[\neg_j\Box_j \Rightarrow]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta / \neg_j A \Rightarrow]}{\mathfrak{N}[\neg_j\Box_j A, \Gamma \Rightarrow \Delta]} \qquad [\Rightarrow \neg_j\Box_j]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg_j\Box_j A/(\Theta \Rightarrow \Lambda, \neg_j A/X)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg_j\Box_j A/(\Theta \Rightarrow \Lambda/X)]}$$

$$[\neg_j\Diamond_j \Rightarrow]\ \frac{\mathfrak{N}[\neg_k\Diamond_j A, \Gamma \Rightarrow \Delta/(\neg_j A, \Theta \Rightarrow \Lambda/X)]}{\mathfrak{N}[\neg_j\Diamond_j A, \Gamma \Rightarrow \Delta/(\Theta \Rightarrow \Lambda/X)]} \qquad [\Rightarrow \neg_j\Box_j]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/ \Rightarrow \neg_j A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg_j\Diamond_j A]}$$

The kj-negated modal rules:

$$[\neg_k\Box_j \Rightarrow]\ \frac{\mathfrak{N}[\neg_k\Box_j A, \Gamma \Rightarrow \Delta/(\neg_k A, \Theta \Rightarrow \Lambda/X)]}{\mathfrak{N}[\neg_k\Box_j A, \Gamma \Rightarrow \Delta/(\Theta \Rightarrow \Lambda/X)]} \qquad [\Rightarrow \neg_k\Box_j]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/ \Rightarrow \neg_k A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg_k\Box_j A]}$$

$$[\neg_k\Diamond_j \Rightarrow]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/\neg_k A \Rightarrow]}{\mathfrak{N}[\neg_k\Diamond_j A, \Gamma \Rightarrow \Delta]} \qquad [\Rightarrow \neg_k\Diamond_j]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg_k\Diamond_j A/(\Theta \Rightarrow \Lambda, \neg_k A/X)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg_k\Diamond_j A/(\Theta \Rightarrow \Lambda/X)]}$$

As an example, we give the rules for $\rhd_j$:

$$[\Rightarrow \rhd_{j_L}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/A \Rightarrow]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \rhd_j A]} \qquad [\Rightarrow \rhd_{j_R}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/ \Rightarrow A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \rhd_j A]}$$

$$[\rhd_j \Rightarrow]\ \frac{\mathfrak{N}[\rhd_j A, \Gamma \Rightarrow \Delta/(A, \Theta \Rightarrow \Lambda/X)] \quad \mathfrak{N}[\rhd_j A, \Gamma \Rightarrow \Delta/(\Xi \Rightarrow \Pi, A/Y)]}{\mathfrak{N}[\rhd_j A, \Gamma \Rightarrow \Delta/(\Theta \Rightarrow \Lambda/X); (\Xi \Rightarrow \Pi/Y)]}$$

$$[\Rightarrow \neg_j\rhd_j]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg_j\rhd_j A/(\neg_j A, \Theta \Rightarrow \Lambda/X)] \quad \mathfrak{N}[\Gamma \Rightarrow \Delta, \neg_j\rhd_j A/(\Xi \Rightarrow \Pi, \neg_j A/Y)]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg_j\rhd_j A/(\Theta \Rightarrow \Lambda/X); (\Xi \Rightarrow \Pi/Y)]}$$

$$[\neg_j\rhd_j \Rightarrow_L]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/\neg_j A \Rightarrow]}{\mathfrak{N}[\neg_j\rhd_j A, \Gamma \Rightarrow \Delta]} \qquad [\neg_j\rhd_j \Rightarrow_R]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/ \Rightarrow \neg_j A]}{\mathfrak{N}[\neg_j\rhd_j A, \Gamma \Rightarrow \Delta]}$$

$$[\Rightarrow \neg_k\rhd_{j_L}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/\neg_k A \Rightarrow]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg_k\rhd_j A]} \qquad [\Rightarrow \neg_k\rhd_{j_R}]\ \frac{\mathfrak{N}[\Gamma \Rightarrow \Delta/ \Rightarrow \neg_k A]}{\mathfrak{N}[\Gamma \Rightarrow \Delta, \neg_k\rhd_j A]}$$

$$[\neg_k\rhd_j \Rightarrow]\ \frac{\mathfrak{N}[\neg_k\rhd_j A, \Gamma \Rightarrow \Delta/(\neg_k A, \Theta \Rightarrow \Lambda/X)] \quad \mathfrak{N}[\neg_k\rhd_j A, \Gamma \Rightarrow \Delta/(\Xi \Rightarrow \Pi, \neg_k A/Y)]}{\mathfrak{N}[\rhd_j A, \Gamma \Rightarrow \Delta/(\Theta \Rightarrow \Lambda/X); (\Xi \Rightarrow \Pi/Y)]}$$

The rules for other non-standard modalities can be found in an analogous fashion.

Theorem 180. *Let* $\clubsuit \in \{\Box, \Diamond, \rhd, \blacktriangleright, \circ, \bullet, \tilde{\circ}, \tilde{\bullet}, \sim, \dot{\sim}\}$ *and* $\mathbf{X}_1, \ldots \mathbf{X}_m \in \{\mathbf{T}, \mathbf{D}, \mathbf{4}, \mathbf{B}\}$. *For any nested sequent* $\mathfrak{N}$, *if* $\mathbb{N}\$\mathbf{MML}_n^{\mathbf{KX}_1,\ldots\mathbf{X}_m^\clubsuit} \vdash \mathfrak{N}$, *then* $\mathbf{MML}_n^{\mathbf{KX}_1,\ldots\mathbf{X}_m^\clubsuit} \models \mathfrak{N}$.

Proof. Similarly to Theorem 27. □

Theorem 181. *Let* $\clubsuit \in \{\Box, \Diamond, \rhd, \blacktriangleright, \circ, \bullet, \tilde{\circ}, \tilde{\bullet}, \sim, \dot{\sim}\}$. *For any nested sequent* $\mathfrak{N}$, *if* $\mathbf{MML}_n^{\mathbf{KX}_1,\ldots\mathbf{X}_m^\clubsuit} \models \mathfrak{N}$, *then* $\mathbb{N}\$\mathbf{MML}_n^{\mathbf{KX}_1,\ldots\mathbf{X}_m^\clubsuit} \vdash \mathfrak{N}$.

Proof. Similarly to Theorems 36 and 174. □

Theorem 182 (Constructive elimination of cut)**.** *Let* $\clubsuit \in \{\Box, \Diamond, \rhd, \blacktriangleright, \circ, \bullet, \tilde{\circ}, \tilde{\bullet}, \sim, \dot{\sim}\}$ *and* $\mathbf{X}_1, \ldots \mathbf{X}_m \in \{\mathbf{T}, \mathbf{D}, \mathbf{4}, \mathbf{B}\}$. *Any derivation* $\mathfrak{D}$ *in* $\mathbb{N}\$\mathbf{MML}_n^{\mathbf{KX}_1,\ldots\mathbf{X}_m^\clubsuit} \vdash \mathfrak{N}$ *can be effectively transformed into a derivation* $\mathfrak{D}'$, *where there is no application of the rule of cut.*

Proof. Similarly to Theorem 41. □

Let us describe natural deduction systems for modal multilattice logics. For the sake of simplicity, we restrict the language $\mathscr{L}_M$: we do not consider (co)implications and suppose that only one type of modalities is in the language (either necessity or possibility for $\mathbf{MML}_n^{\mathbf{S5}}$ and only necessity for $\mathbf{MML}_n^{\mathbf{S4}}$). Our rules for modalities are based on Biermann and de Paiva's rules [16] and their Kürbis' [100] adaptation for **S5**, rules for other connectives are based on Milne's [125] natural deduction system for classical logic.

The non-negated rules:

$$
(\wedge_j I)^a \ \dfrac{\begin{array}{ccc} & & [A \wedge_j B]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ A & B & C \end{array}}{C}
\qquad
(\wedge_j E)^{a,b} \ \dfrac{\begin{array}{cc} & [A]^a[B]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ A \wedge_j B & C \end{array}}{C}
$$

$$
(\vee_j I_1)^a \ \dfrac{\begin{array}{cc} & [A \vee_j B]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ A & C \end{array}}{C}
\qquad
(\vee_j I_2)^a \ \dfrac{\begin{array}{cc} & [A \vee_j B]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ B & C \end{array}}{C}
\qquad
(\vee_j E)^{a,b} \ \dfrac{\begin{array}{ccc} & [A]^a & [B]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ A \vee_j B & C & C \end{array}}{C}
$$

The jj-negated rules:

$$
(\neg_j \wedge_j I_1)^a \ \dfrac{\begin{array}{cc} & [\neg_j(A \wedge_j B)]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_j A & C \end{array}}{C}
\qquad
(\neg_j \wedge_j I_2)^a \ \dfrac{\begin{array}{cc} & [\neg_j(A \wedge_j B)]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_j B & C \end{array}}{C}
$$

$$
(\neg_j \wedge_j E)^{a,b} \ \dfrac{\begin{array}{ccc} & [\neg_j A]^a & [\neg_j B]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg_j(A \wedge_j B) & C & C \end{array}}{C}
$$

$$
(\neg_j \vee_j I)^a \ \dfrac{\begin{array}{ccc} & & [\neg_j(A \vee_j B)]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg_j A & \neg_j B & C \end{array}}{C}
\qquad
(\neg_j \vee_j E)^{a,b} \ \dfrac{\begin{array}{cc} & [\neg_j A]^a[\neg_j B]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_j(A \vee_j B) & C \end{array}}{C}
$$

$$
(\neg_j\neg_j I)^a \ \dfrac{\begin{array}{cc} & [\neg_j\neg_j A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ A & C \end{array}}{C}
\qquad
(\neg_j\neg_j E)^a \ \dfrac{\begin{array}{cc} & [A]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_j\neg_j A & C \end{array}}{C}
$$

The kj-negated rules:

$$
(\neg_k \wedge_j I)^a \ \dfrac{\begin{array}{ccc} & & [\neg_k(A \wedge_j B)]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 & \mathfrak{D}_3 \\ \neg_k A & \neg_k B & C \end{array}}{C}
\qquad
(\neg_k \wedge_j E)^{a,b} \ \dfrac{\begin{array}{cc} & [\neg_k A]^a[\neg_k B]^b \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_k(A \wedge_j B) & C \end{array}}{C}
$$

$$
(\neg_k \vee_j I_1)^a \ \dfrac{\begin{array}{cc} & [\neg_k(A \vee_j B)]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_k A & C \end{array}}{C}
\qquad
(\neg_k \vee_j I_2)^a \ \dfrac{\begin{array}{cc} & [\neg_k(A \vee_j B)]^a \\ \mathfrak{D}_1 & \mathfrak{D}_2 \\ \neg_k B & C \end{array}}{C}
$$

$$(\neg_k \vee_j E)^{a,b}\ \dfrac{\begin{matrix}\mathfrak{D}_1\\ \neg_k(A \vee_j B)\end{matrix}\quad \begin{matrix}[\neg_k A]^a\\ \mathfrak{D}_2\\ C\end{matrix}\quad \begin{matrix}[\neg_k B]^b\\ \mathfrak{D}_3\\ C\end{matrix}}{C}$$

$$(\neg_j\neg_k I)\ \dfrac{\begin{matrix}[A]^a\\ \mathfrak{D}_1\\ C\end{matrix}\quad \begin{matrix}[\neg_j\neg_k A]^b\\ \mathfrak{D}_2\\ C\end{matrix}}{C} \qquad (\neg_j\neg_k E)\ \dfrac{\begin{matrix}\mathfrak{D}_1\\ \neg_j\neg_k A\end{matrix}\quad \begin{matrix}\mathfrak{D}_2\\ A\end{matrix}}{C}$$

The non-negated modal rules:

$$(\Box_j GI)^{a,b}\ \dfrac{\begin{matrix}[\Box_j A]^a\\ \mathfrak{D}_0\\ F\end{matrix}\quad \begin{matrix}\mathfrak{D}_1 \dots \mathfrak{D}_m\\ B_1 \dots B_m\end{matrix}\quad \begin{matrix}[B_1 \dots B_m]^b\\ \mathfrak{D}\\ A\end{matrix}}{F} \qquad (\Box_j GE)^a\ \dfrac{\begin{matrix}\mathfrak{D}_1\\ \Box_j A\end{matrix}\quad \begin{matrix}[A]^a\\ \mathfrak{D}_2\\ F\end{matrix}}{F}$$

$$(\Diamond_j GI)^a\ \dfrac{\begin{matrix}[\Diamond_j A]^a\\ \mathfrak{D}_1\\ F\end{matrix}\quad \begin{matrix}\mathfrak{D}_2\\ A\end{matrix}}{F} \qquad (\Diamond_j E)^a\ \dfrac{\begin{matrix}\mathfrak{D}_0\\ \Diamond_j A\end{matrix}\quad \begin{matrix}\mathfrak{D}_1 \dots \mathfrak{D}_m\\ B_1 \dots B_m\end{matrix}\quad \begin{matrix}[A, B_1 \dots B_m]^a\\ \mathfrak{E}\\ C\end{matrix}}{C}$$

where $B_1, \dots, B_m$ are exactly the undischarged assumptions in $\mathfrak{D}$ and $A, B_1, \dots, B_m$ in $\mathfrak{E}$. For $\mathbf{MML}_n^{\mathbf{S4}}$, B_i $(1 \leqslant i \leqslant n)$ are required to be of the form $\Box_j D_i$ or $\neg_k\Box_j D_i$ and C to be of the form $\Diamond_j D$ or $\neg_k\Diamond_j D$. For $\mathbf{MML}_n^{\mathbf{S5}}$, $A, B_1, \dots, B_m, C$ are required to be modalized.

The jj-negated modal rules:

$$(\neg_j\Box_j GI)^a\ \dfrac{\begin{matrix}[\neg_j\Box_j A]^a\\ \mathfrak{D}_1\\ F\end{matrix}\quad \begin{matrix}\mathfrak{D}_2\\ \neg_j A\end{matrix}}{F} \qquad (\neg_j\Box_j E)^a\ \dfrac{\begin{matrix}\mathfrak{D}_0\\ \neg_j\Box_j A\end{matrix}\quad \begin{matrix}\mathfrak{D}_1 \dots \mathfrak{D}_m\\ B_1 \dots B_m\end{matrix}\quad \begin{matrix}[\neg_j A, B_1 \dots B_m]^a\\ \mathfrak{E}\\ C\end{matrix}}{C}$$

$$(\neg_j\Diamond_j GI)^{a,b}\ \dfrac{\begin{matrix}[\neg_j\Diamond_j A]^a\\ \mathfrak{D}_0\\ F\end{matrix}\quad \begin{matrix}\mathfrak{D}_1 \dots \mathfrak{D}_m\\ B_1 \dots B_m\end{matrix}\quad \begin{matrix}[B_1 \dots B_m]^b\\ \mathfrak{D}\\ \neg_j A\end{matrix}}{F} \qquad (\neg_j\Diamond_j GE)^a\ \dfrac{\begin{matrix}\mathfrak{D}_1\\ \neg_j\Diamond_j A\end{matrix}\quad \begin{matrix}[\neg_j A]^a\\ \mathfrak{D}_2\\ F\end{matrix}}{F}$$

where $B_1, \dots, B_m$ are exactly the undischarged assumptions in $\mathfrak{D}$ and $\neg_j A, B_1, \dots, B_m$ in $\mathfrak{E}$. For $\mathbf{MML}_n^{\mathbf{S4}}$, B_i $(1 \leqslant i \leqslant n)$ are required to be of the form $\Box_j D_i$ or $\neg_k\Box_j D_i$ and C to be of the form $\Diamond_j D$ or $\neg_k\Diamond_j D$. For $\mathbf{MML}_n^{\mathbf{S5}}$, $\neg_j A, B_1, \dots, B_m, C$ are required to be modalized.

The kj-negated modal rules:

$$(\neg_k\Box_j GI)^{a,b}\ \dfrac{\begin{matrix}[\neg_k\Box_j A]^a\\ \mathfrak{D}_0\\ F\end{matrix}\quad \begin{matrix}\mathfrak{D}_1 \dots \mathfrak{D}_m\\ B_1 \dots B_m\end{matrix}\quad \begin{matrix}[B_1 \dots B_m]^b\\ \mathfrak{D}\\ \neg_k A\end{matrix}}{F} \qquad (\neg_k\Box_j GE)^a\ \dfrac{\begin{matrix}\mathfrak{D}_1\\ \neg_k\Box_j A\end{matrix}\quad \begin{matrix}[\neg_k A]^a\\ \mathfrak{D}_2\\ F\end{matrix}}{F}$$

$$(\neg_k\Diamond_j GI)^a\ \dfrac{\begin{matrix}[\neg_k\Diamond_j A]^a\\ \mathfrak{D}_1\\ F\end{matrix}\quad \begin{matrix}\mathfrak{D}_2\\ \neg_k A\end{matrix}}{F} \qquad (\neg_k\Diamond_j E)^a\ \dfrac{\begin{matrix}\mathfrak{D}_0\\ \neg_k\Diamond_j A\end{matrix}\quad \begin{matrix}\mathfrak{D}_1 \dots \mathfrak{D}_m\\ B_1 \dots B_m\end{matrix}\quad \begin{matrix}[\neg_k A, B_1 \dots B_m]^a\\ \mathfrak{E}\\ C\end{matrix}}{C}$$

where $B_1, \dots, B_m$ are exactly the undischarged assumptions in $\mathfrak{D}$ and $\neg_k A, B_1, \dots, B_m$ in $\mathfrak{E}$. For $\mathbf{MML}_n^{\mathbf{S4}}$, B_i $(1 \leqslant i \leqslant n)$ are required to be of the form $\Box_j D_i$ or $\neg_k\Box_j D_i$ and C to be of the form $\Diamond_j D$ or $\neg_k\Diamond_j D$. For $\mathbf{MML}_n^{\mathbf{S5}}$, $\neg_k A, B_1, \dots, B_m, C$ are required to be modalized.

The notion of a deduction is defined in a standard way.

Theorem 183. *Let* $\mathbf{L} \in \{\mathbf{MML}_n^{\mathbf{S5}}, \mathbf{MML}_n^{\mathbf{S4}}\}$. *For every finite set of* $\mathscr{L}_M$*-formulas* Γ *and every* $\mathscr{L}_M$*-formula* A, *it holds that* $\Gamma \models_{\mathbf{L}} A$ *iff* $\Gamma \vdash_{\mathbf{L}} A$ *in a natural deduction system for* $\mathbf{L}$.

Proof. By induction on the height of the derivation, one can show that $\Gamma \vdash_{\mathbf{L}} A$ in a natural deduction system for $\mathbf{L}$ iff $\Gamma \vdash_{\mathbf{L}} A$ in a hypersequent (nested) sequent calculus for $\mathbf{L}$. By Theorems 173, 174, 180, 181, $\Gamma \vdash_{\mathbf{L}} A$ in a hypersequent (nested) sequent calculus for $\mathbf{L}$ iff $\Gamma \models_{\mathbf{L}} A$. □

Our normalisation proof is an adaptation of Kürbis' proof [103] for Milne's [125] natural deduction system for classical logic.

The notion of a maximal formula is understood according to Definition 95. A degree of a formula is understood in the same way as the complexity of a formula in the constructive proof of cut elimination for $\mathbf{MML}_n^{\mathbf{S5}}$.

Definition 184 (Segment)**.** [103, Definitions 5, 6, the notation adjusted] A *segment* is a sequence of formula occurrences $C_1 ... C_m$ in a deduction such that either

(i) $m > 1$ and for all $i < m$, C_i is an arbitrary premise of an application of a rule and C_{i+1} is its conclusion, and C_n is not an arbitrary premise of an application of a rule; or

(ii) $m \geqslant 1$ and C_1 is the conclusion of $(\neg_k\neg_j E)$ and for all $i < m$, C_i is an arbitrary premise of an application of a rule and C_{i+1} is its conclusion, and C_n is not an arbitrary premise of an application of a rule.

The length of a segment is the number of formula occurrences of which it consists, its degree the degree of any such formula.

A *maximal segment* is a segment the last formula of which is the major premise of an elimination rule.

Every conclusion of the rule $(\neg_k\neg_j)$ forms part of a segment of length 1.

The notions of a deduction in normal form and a rank of a deduction are understood according to Definitions 54 and 55.

At the first stage of a proof of the normalisation theorem for $\mathbf{MML}_n^{\mathbf{S5}}$ and $\mathbf{MML}_n^{\mathbf{S4}}$ (we prove the theorem for both logics simultaneously due to the formulation of modal rules) we need to observe that any application of a general introduction rule can be transformed into one that discharges exactly one major assumption. It can be done in the same way as in Lemma 98. In this proof we apply "*unique discharge convention* on introduction rules: every application of an introduction rule for $*$ discharges exactly one formula occurrence of the form $A * B$ or $*A$." [103, p. 115]

Let us present reduction procedures for maximal formulas.

The maximal formula is of the form $\neg_j(A \vee_j B)$. Convert the deduction on the left into the deduction on the right:

$$\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \neg_j A\end{array} \qquad \begin{array}{c}\mathfrak{D}_2\\ \neg_j B\end{array} \qquad \begin{array}{c} \dfrac{[\neg_j(A\vee_j B)]^a \qquad \begin{array}{c}[\neg_j A]^b[\neg_j B]^c\\ \mathfrak{D}_3\\ C\end{array}}{C}\,(\neg_j\vee_j E)^{b,c}\\ \mathfrak{D}_4\\ D\end{array}}{D}\,(\neg_j\vee_j I)^a \quad \dashrightarrow \quad \begin{array}{c}\mathfrak{D}_1 \qquad \mathfrak{D}_2\\ \neg_j A \qquad \neg_j B\\ \mathfrak{D}_3\\ C\\ \mathfrak{D}_4\\ D\end{array}$$

The maximal formula is of the form $\neg_j\neg_k A$. Convert the deduction on the left into the deduction on the right:

$$\dfrac{\begin{array}{c}[A]^a\\ \mathfrak{D}_1\\ C\end{array} \qquad \begin{array}{c}\dfrac{[\neg_j\neg_k A]^b \qquad \begin{array}{c}\mathfrak{D}_2\\ A\end{array}}{B}\,(EFQ)\\ \mathfrak{D}_3\\ C\end{array}}{C}\,(EM)^{a,b} \quad \dashrightarrow \quad \begin{array}{c}\mathfrak{D}_2\\ A\\ \mathfrak{D}_1\\ C\\ \mathfrak{D}_3^*\\ C\end{array}$$

where $\mathfrak{D}_3^*$ is is defined via the ρ-reduction described in the normalisation proof for Segerberg's system on p. 55.

The other cases are considered similarly. Let us give a list of them:[4]

- the maximal formula is of the form $A \wedge_j B$,
- the maximal formula is of the form $A \vee_j B$,
- the maximal formula is of the form $\neg_j(A \wedge_j B)$,
- the maximal formula is of the form $\neg_k(A \wedge_j B)$,
- the maximal formula is of the form $\neg_k(A \vee_j B)$,
- the maximal formula is of the form $\Box_j A$,
- the maximal formula is of the form $\neg_j\Box_j A$,
- the maximal formula is of the form $\neg_k\Box_j A$.[5]

Let us present permutative reduction procedures for maximal segments.

A major premise of an elimination rule is concluded by $(\neg_j\neg_k E)$. Then remove the application of the elimination rule and use $(\neg_j\neg_k E)$ to conclude the discharged assumption(s) of (one of) the side deductions concluding with an arbitrary premise instead. As an example, consider the following case:

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \neg_j\neg_k A\end{array} \qquad \begin{array}{c}\mathfrak{D}_2\\ A\end{array}}{\neg_k\Box_j A} \qquad \begin{array}{c}[\neg_k A]^a\\ \mathfrak{D}_3\\ F\end{array}}{F} \quad \dashrightarrow \quad \begin{array}{c}\dfrac{\begin{array}{c}\mathfrak{D}_1\\ \neg_j\neg_k A\end{array} \qquad \begin{array}{c}\mathfrak{D}_2\\ A\end{array}}{\neg_k A}\\ \mathfrak{D}_3\\ F\end{array}$$

The major premise of $(\neg_j\Box_j E)$ is derived by $(\neg_k \vee_j I_1)$.

$$\dfrac{\dfrac{\begin{array}{c}\mathfrak{E}_1\\ \neg_k A\end{array} \qquad \begin{array}{c}[\neg_k(A \vee_j B)]^a\\ \mathfrak{E}_2\\ \neg_j\Box_j A\end{array}}{\neg_j\Box_j A}\,a \qquad \begin{array}{c}\mathfrak{D}_1 \ldots \mathfrak{D}_m\\ B_1 \ldots B_m\end{array} \qquad \begin{array}{c}[\neg_j A, B_1 \ldots B_m]^b\\ \mathfrak{H}_1\\ C\\ \mathfrak{H}_2\\ E\end{array}}{E}\,b$$

where $\neg_j A, B_1, \ldots, B_m$ are exactly the undischarged assumptions in $\mathfrak{H}_1$. For $\mathbf{MML}_n^{\mathbf{S4}}$, B_i is required to be of the form $\Box_j D_i$ or $\neg_k\Box_j D_i$ $(1 \leqslant i \leqslant m)$ and C to be of the form $\neg_j\Box_j D$. For $\mathbf{MML}_n^{\mathbf{S5}}$, $A, B_1, \ldots, B_m, C$ are required to be modalized or negatively modalized.

We change the order of the applications of the rules as follows:

$$\dfrac{\begin{array}{c}\mathfrak{E}_1\\ \neg_k A\end{array} \qquad \dfrac{\begin{array}{c}[\neg_k(A \vee_j B)]^a\\ \mathfrak{E}_2\\ \neg_j\Box_j A\end{array} \qquad \begin{array}{c}\mathfrak{D}_1 \ldots \mathfrak{D}_m\\ B_1 \ldots B_m\end{array} \qquad \begin{array}{c}[\neg_j A, B_1 \ldots B_m]^b\\ \mathfrak{H}_1\\ C\end{array} \qquad \begin{array}{c}[C]^c\\ \mathfrak{H}_2\\ E\end{array}}{E}\,b,c}{E}\,a$$

[4]Actually, the cases with $A \wedge_j B$ and $A \vee_j B$ coincide with the cases for $A \wedge B$ and $A \vee B$ from [103].

[5]If one considers the language with $\Diamond_j$ instead of $\Box_j$, then one should consider analogous cases with maximal formulas of the forms $\Diamond_j A$, $\neg_j\Diamond_j A$, and $\neg_k\Diamond_j A$; these cases are treated analogously too.

where $\neg_j A, B_1, \ldots, B_m$ are exactly the undischarged assumptions in $\mathfrak{H}_1$. For $\mathbf{MML}_n^{\mathbf{S4}}$, B_i is required to be of the form $\Box_j D_i$ or $\neg_k\Box_j D_i$ $(1 \leqslant i \leqslant m)$ and C to be of the form $\neg_j\Box_j D$. For $\mathbf{MML}_n^{\mathbf{S5}}$, $\neg_j A, B_1, \ldots, B_m, C$ are required to be modalized or negatively modalized.

The major premise of $(\neg_j\Box_j E)$ is derived by $(\neg_j\Box_j E)$.

$$\dfrac{\dfrac{\begin{matrix}\\ \mathfrak{E}_0\\ \neg_j\Box_j E\end{matrix}\quad \begin{matrix}\\ \mathfrak{E}_1\ldots\mathfrak{E}_l\\ F_1\ldots F_l\end{matrix}\quad \begin{matrix}[\neg_j E, F_1\ldots F_l]^a\\ \mathfrak{H}_1\\ G\end{matrix}\quad \begin{matrix}[G]^c\\ \mathfrak{H}_3\\ \neg_j\Box_j A\end{matrix}}{\neg_j\Box_j A}\,{\scriptstyle a,c}\quad \begin{matrix}\mathfrak{D}_1\ldots\mathfrak{D}_m\\ B_1\ldots B_m\end{matrix}\quad \begin{matrix}[\neg_j A, B_1\ldots B_m]^b\\ \mathfrak{H}_2\\ C\end{matrix}\quad \begin{matrix}[C]^d\\ \mathfrak{H}_4\\ I\end{matrix}}{I}\,{\scriptstyle b,d}$$

where $\neg_j E, F_1 \ldots F_l$ and $\neg_j A, B_1, \ldots, B_m$ are exactly the undischarged assumptions in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. For $\mathbf{MML}_n^{\mathbf{S4}}$, B_i is required to be of the form $\Box_j D_i$ or $\neg_k\Box_j D_i$, F_i is required to be of the form $\Box_j H_i$ or $\neg_k\Box_j H_i$ $(1 \leqslant i \leqslant m)$, C to be of the form $\neg_j\Box_j D$, G to be of the form $\neg_j\Box_j J$. For $\mathbf{MML}_n^{\mathbf{S5}}$, $\neg_j A, B_1, \ldots, B_m, C, \neg_j E, F_1 \ldots F_l, G$ are required to be modalized or negatively modalized.

We change the order of the applications of the rules as follows:

$$\dfrac{\begin{matrix}\\ \mathfrak{E}_0\\ \neg_j\Box_j E\end{matrix}\quad \begin{matrix}\\ \mathfrak{E}_1\ldots\mathfrak{E}_l\\ F_1\ldots F_l\end{matrix}\quad \begin{matrix}[\neg_j E, F_1\ldots F_l]^a\\ \mathfrak{H}_1\\ G\end{matrix}\quad \dfrac{\begin{matrix}[G]^c\\ \mathfrak{H}_3\\ \neg_j\Box_j A\end{matrix}\quad \begin{matrix}\mathfrak{D}_1\ldots\mathfrak{D}_m\\ B_1\ldots B_m\end{matrix}\quad \begin{matrix}[\neg_j A, B_1\ldots B_m]^b\\ \mathfrak{H}_2\\ C\end{matrix}\quad \begin{matrix}[C]^d\\ \mathfrak{H}_4\\ I\end{matrix}}{I}\,{\scriptstyle b,d}}{I}\,{\scriptstyle a,c}$$

where $\neg_j E, F_1 \ldots F_l$ and $\neg_j A, B_1, \ldots, B_m$ are exactly the undischarged assumptions in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. For $\mathbf{MML}_n^{\mathbf{S4}}$, B_i is required to be of the form $\Box_j D_i$ or $\neg_k\Box_j D_i$, F_i is required to be of the form $\Box_j H_i$ or $\neg_k\Box_j H_i$ $(1 \leqslant i \leqslant m)$, C to be of the form $\neg_j\Box_j D$, G to be of the form $\neg_j\Box_j J$. For $\mathbf{MML}_n^{\mathbf{S5}}$, $\neg_j A, B_1, \ldots, B_m, C, \neg_j E, F_1 \ldots F_l, G$ are required to be modalized or negatively modalized.

The other cases are considered similarly.

Theorem 185. *Any deduction in* $\mathbf{MML}_n^{\mathbf{S5}}$ *or* $\mathbf{MML}_n^{\mathbf{S4}}$ *can be converted into a deduction in normal form.*

Proof. By induction over the rank of deductions. Similarly to Theorem 56. □

Corollary 186. *If* $\mathfrak{D}$ *is a deduction in normal form, then all major premises of elimination rules are (discharged or undischarged) assumptions of* $\mathfrak{D}$*.*

Theorem 187. *Deductions in normal forms in in* $\mathbf{MML}_n^{\mathbf{S5}}$ *or* $\mathbf{MML}_n^{\mathbf{S4}}$ *have the negation subformula property.*

Proof. By inspection of the rules and an induction over the order of branches. □

Chapter 5

Conclusion

This section is devoted to two issues: summing up the results presented in this work and pointing out the possible routes of future research arising from the obtained results.

The following results were described in the thesis:

1. We have developed cut-free hypersequent (the case of **S5**-based logics) and nested sequent calculi (the case of weaker logics) for modal logics with non-standard modalities (such as (non)contingency, essence, accidence, and negated modalities). The completeness proof has been established by a Hintikka-style argument. As a consequence, the cut admissibility theorem has been obtained. Besides, syntactic constructive cut admissibility proof by Metcalfe, Olivetti, and Gabbay's strategy [123] has been given. This result is published in [148].

2. We have proved the normalisation theorem for classical propositional logic formulated in the language with at least one n-ary operator (Segerberg's [173] natural deduction system) as well as its modal extensions (**S5** and **S4** with necessity, possibility, and negated modalities being formulated via general introduction and elimination rules). The standard subformula property has been established.

3. We have generalised natural deduction systems from [145] for three-valued logics: first, instead of unary and binary connectives as in [145] we have considered n-ary ones; second, we have shown that the approach developed in [145] works not only for Łukasiewicz's, Heyting's, and Bochvar's negations, but some other ones described by Omori and Wansing in [141], including Post's negation [159] and its converse [147]. We have proved the normalisation theorem for all the three-valued logics in question. The negation subformula property has been established. This result is an extended version of a joint work with Nils Kürbis [106] which is now under review.

4. We have transformed the above-mentioned natural deduction systems into cut-free sequent calculi; it has been shown that the resulting sequent calculi are equivalent to the original natural deduction systems. Then we have justified their completeness together with the cut admissibility theorem by Hintikka-style argument and given a constructive cut admissibility argument by Metcalfe, Olivetti, and Gabbay's strategy.

5. By the same method, we have provided a constructive cut admissibility proof for Kooi and Tamminga's sequent calculi [97] (in their paper, only semantic cut admissibility had been given) for four-valued logics (n-ary extensions of the negation fragment of **FDE**). We have shown that their approach can be built on the basis of the negation fragments of other four-valued logics; we have considered the negations studied by Omori and Wansing in [141]. For these additional logics, the constructive proof of the cut admissibility theorem and a Hintikka-style completeness proof have been given.

6. On the basis of Kooi and Tamminga's sequent calculi for four-valued logics [97] we have introduced natural deduction systems for the same logics. We have proved the normalisation theorem for them. The negation subformula property has been established.

7. We have supplied all the three- and four-valued logics in question with many-valued modalities (**S5**-style and weaker ones; both standard and non-standard ones). Cut-free hypersequent and nested sequent calculi have been developed. Hintikka-style completeness and constructive cut admissibility have been given for them, and the negation subformula property has been established. The natural deduction systems for the **S5**-style modal many-valued logics with standard and negated modalities have been given with the proof of the normalisation theorem.

8. Last but not least, we have investigated modal multilattice logics, an algebraic generalisation of Belnap-Dunn-style four-valued modal logics. Cut-free hypersequent and nested sequent calculi have been given. Due to the machinery of nested sequent calculi we have managed to consider more modal multilattice logics (based on normal modal logics) than is usually done in the literature on this topic [92, 71, 69]. In the joint works by the author and Grigoriev [71, 69] the main results, such as Kripke completeness and cut admissibility, were obtained via embedding functions. Here we have given new proofs for these results: by Hintikka-style argument. Besides, we have presented a constructive cut admissibility proof based on Metcalfe, Olivetti, and Gabbay's strategy. We have developed natural deduction systems for multilattice logic and some of its modal extensions: the normalisation theorem and the negation subformula property have been established.

The following topics may become the subject of future research arising from the above-described results:

1. The formulation of the natural deduction systems for **S5**-modal logics with non-standard modalities different from the negated ones: non(contingency), essence, and accident. The development of the natural deduction systems for modal logics with non-standard modalities based on the logics weaker than **S5**. The establishing of the normalisation theorem and subformula property for them.

2. An analogous task can be formulated for many-valued and multilattice modal logics.

3. An adaptation of correspondence analysis for non-deterministic logics, in particular for the logics with non-deterministic negations investigated by Omori and Wansing [141].

4. A generalisation of the technique of correspondence analysis for the case of k-valued logics.

5. An investigation of first-order extensions of the logics considered in this text. In particular, one may explore the systems of neutral free logic on their basis, which can be useful for the development of the theory of definite descriptions.

6. A formulation and investigation of proof-searching algorithms for the considered logics, and even the development of a computer programme on their basis automatically building proofs in the proof systems in question. Notice that in the papers [149, 150, 151] Shangin and the author of this text have explored sound, complete, and terminating proof-searching algorithms for obtained via correspondence analysis natural deduction systems for some many-valued logics. Although all the logics considered in this thesis are known to be decidable, one may develop, on the basis of the presented proof systems, decidability algorithms for these logics.

7. The development of proof-theoretic semantics for the logics for which we have proved normalisation and the (negation) subformula property.

8. The development of analytic tableaux for the logics considered in this text. One may think also about other less popular types of calculi, such as synthetic tableaux, resolution, or Rasiowa-Sikorski's calculi.

9. On the basis of the established cut admissibility and subformula property theorems, one may try to prove the interpolation theorem, Beth's definability property, and Maksimova's separation property for the logics in question.

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