Professor Alexandru DIMCA

LABORATOIRE J.A. DIEUDONNÉ, UMR du CNRS 7351
Université Côte d'Azur, Faculté des Sciences
Parc Valrose, 06108 Nice Cedex 02, FRANCE.
tel: +33-622-722-582, email: dimca@unice.fr

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Recommendation letter for Dr. Justyna Szpond

It is my pleasure to write this recommendation letter since I have a very high opinion of the quality of Justyna Szpond's work. Szpond's main results are substantial contributions to Commutative Algebra and Algebraic Geometry. They can be divided into the following two areas.

- *The containment problem*

Let $R = K[x_0, \ldots, x_N]$ be the polynomial ring in $N+1$ indeterminates $x_0, \ldots, x_N$ with coefficients in a field K. For a homogeneous ideal $I \subset R$, one can consider two descending infinite series of ideals, namely the powers of the ideal I

$$R \supset I \supset I^2 \supset \ldots \supset I^m \supset \ldots$$

and the symbolic powers of the ideal I

$$R \supset I \supset I^{(2)} \supset \ldots \supset I^{(m)} \supset \ldots,$$

where

$$I^{(m)} = \cap_{P \in Ass(I)}(R \cap (I^m)_P).$$

The **Containment Problem** asks for which pairs of positive integers m, r one has the inclusion

$$I^{(m)} \subset I^r$$

for all proper homogeneous ideals $I \subset R$. A celebrated theorem, due to Ein-Lazarsfeld-Smith and independently to Hochster-Huneke says that the above inclusion holds for all $m \geq Nr$. When the zero-set $Z(I)$ consists of finitely many points, it was conjectured that a stronger result holds, namely that the above inclusion holds for all

$$m \geq Nr - N + 1.$$

The first interesting case of this conjecture is when $N = r = 2$. The first counter-example over the complex numbers in this case was found in 2013 by Dumnicki, Szemberg and Tutaj-Gasińska. It was for some time an open question whether such counter-examples existed over the real numbers $\mathbb{R}$, or even stronger, over the rational numbers $\mathbb{Q}$. A counter-example over $\mathbb{R}$ was constructed using the Böröczky arrangement of 12 lines, namely the ideal I such that

$$I^{(3)} \not\subset I^2$$

is obtained as the set of polynomials vanishing at the 12 points in $\mathbb{P}^2$, which are dual to the 12 lines in the Böröczky arrangement. In a joint work with Lampa-Baczyńska listed as [Hab7], Justyna Szpond was able to deform the Böröczky arrangement into an arrangement defined over $\mathbb{Q}$, and in this way they obtained a counter-example to the inclusion $I^{(3)} \subset I^2$ over $\mathbb{Q}$.

One can replace points in $\mathbb{P}^2$ by codimension 2 subvarieties in a higher dimensional projective space $\mathbb{P}^N$ and ask a similar question. Using codimension 2 flats coming from Fermat type arrangements defined by

$$F_{N,n} = \prod_{0 \le i < j \le N} (x_i^n - x_j^n) = 0,$$

the corresponding counter-examples have been constructed, in [Hab5] for $N = 3$ and in [Hab6] for all $N \ge 4$, both papers in collaboration with G. Malara.

- *The existence of unexpected hypersurfaces*

The *general problem* here can be stated as follows: given a linear system $W \subset |\mathcal{O}_{\mathbb{P}^N}(d)|$ of hypersurfaces of degree d in the projective space $\mathbb{P}^N$, and given s general points $P_1, \dots, P_s$ in $\mathbb{P}^N$ with assigned multiplicities $m_1, \dots, m_s \in \mathbb{Z}_{>0}$, find the dimension of the subsystem of W consisting of the hypersurfaces having at P_j a multiplicity $\ge m_j$ for all $j = 1, \dots, s$. The only known answers are when $W = |\mathcal{O}_{\mathbb{P}^N}(d)|$ and $s = 1$, or $m_1 = \ldots = m_s \le 2$. It was noted in 2013 by Di Gennaro, Ilardi and Vallès, that when $W \ne |\mathcal{O}_{\mathbb{P}^N}(d)|$, even the case $s = 1$ may be not clear. This question was studied in detail in a paper by Cook II, Harbourne, Migliore and Nagel in 2018, where they introduced the term unexpected hypersurfaces as follows. For a linear system $W \subset |\mathcal{O}_{\mathbb{P}^N}(d)|$, a generic point $P \in \mathbb{P}^N$ and an integer $m \ge 2$, we say that W *contains an unexpected hypersurface* if

$$h^0(\mathbb{P}^N, W \otimes I(P)^m) > \max\left\{\dim W - \binom{N+m-1}{N}, 0\right\},$$

where $I(P)$ is the ideal of functions vanishing at the point P. It is usual to specify a linear system W by fixing a subscheme $Z \subset \mathbb{P}^N$ and taking $W = |\mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z)|$. If this is the case, and if the above inequality holds, then we say that Z *admits an unexpected hypersurface of degree d and multiplicity m*.

Justyna Szpond noticed in the paper [Hab4] that the points Z in $\mathbb{P}^2$ used to define the linear system W by Di Gennaro, Ilardi and Vallès are the dual points of the line arrangement in $\mathbb{P}^2$ given by

$$xyz(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0.$$

Moreover, by the work of Cook II, Harbourne, Migliore and Nagel it was known that the set of points Z dual to the Fermat arrangement

$$F_{2,d} = (x^d - y^d)(y^d - z^d)(z^d - x^d) = 0$$

has an unexpected curve of degree $d + 2$ and multiplicity $m = d + 1$. Justyna Szpond was able to complete this picture by showing that the set of dual point to the arrangement

$$xy(x^d - y^d)(y^d - z^d)(z^d - x^d) = 0 \text{ for } d = 3$$

and

$$x(x^d - y^d)(y^d - z^d)(z^d - x^d) = 0 \text{ for } d = 4$$

also have unexpected curves of degree $d+2$ and multiplicity $m = d+1$. Moreover, explicit equations were provided for all these unexpected curves. These equations, which may be regarded as bihomogeneous polynomials

$$f(a_0, \dots, a_N; x_0, \dots, x_N),$$

where $(a_0 : \dots : a_N)$ are the coordinates of the generic point P, were studied in the paper [Hab2], joint work with T. Bauer, G. Malara and T. Szemberg. The authors noted a certain duality between the two sets of variables (geometrically this relates the unexpected hypersurface to its tangent cone at the singular point P), which was later extensively studied by several mathematicians and it is now known under the name *BMSS duality*, in reference to the names of the authors of the important paper [Hab2].

If one takes Z to be the set of intersection points of the Fermat line arrangement $F_{2,d} = 0$, then it is shown in [Hab1] that this set admits an unexpected curve of degree $d+2$ and multiplicity $m = 4$. This series of unexpected curves, with increasing degree but fixed multiplicity, is very interesting.

The first example of unexpected hypersurfaces of dimension > 1 was constructed in the paper [Hab2]. One starts with the Fermat plane arrangement in $\mathbb{P}^3$ given by $F_{3,3} = 0$ and takes Z to be the set of points of multiplicity ≥ 6 in this arrangement. Then the authors show that this set Z admits an unexpected surface of degree 4 and multiplicity 3, whose equation is given explicitly.

In the article [Hab1] the generic point P with multiplicity m is replaced by a finite set of generic points $P_1, \dots, P_s$ in $\mathbb{P}^N$ with assigned multiplicities $m_1, \dots, m_s$. We say that Z *admits an unexpected hypersurface of degree d and multiplicities* $m_1, \dots, m_s$ if

$$h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z) \otimes I(P_1)^{m_1} \otimes \dots \otimes I(P_s)^{m_s}) >$$
$$> \max\left\{ h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z)) - \binom{N+m_1-1}{N} - \dots - \binom{N+m_s-1}{N}, 0 \right\}.$$

Then Szpond constructs a set of points $Z \subset \mathbb{P}^5$ which admits an unexpected hypersurface of degree $d = 4$ and multiplicities $m_1 = 3$ and $m_2 = 2$.

I am really impressed by Justyna Szpond's ability to master a wide range of problems and of difficult technical tools. Her openness in research areas, his great energy and enthusiasm, and her proven gift of interacting with other mathematicians are also valuable assets. For all these reasons, I strongly support the conferment of the degree of a habilitated doctor to Justyna Szpond.

Alexandru DIMCA